WELL-POSEDNESS AND STABILITY OF THE LAGRANGE REPRESENTATION OF THE N-D WAVE EQUATION VIA BOUNDARY TRIPLES

BERNHARD AIGNER AND NATHANAEL SKREPEK

Abstract. We study the Lagrange representation of the wave equation with generalized Laplacian $\operatorname{div} T\nabla$. We allow the coefficients—the Young modulus T and the density ρ—to be L^∞ or even nonlocal operators. Moreover, the Lipschitz boundary of the domain Ω can be split into several parts admitting Dirichlet, Neumann and/or Robin-boundary conditions of displacement, velocity and stress. We show well-posedness of this classical model of the wave equation utilizing boundary triple theory for skew-adjoint operators. In addition we show semi-uniform stability of solutions under slightly stronger assumptions by means of a spectral result.

1. Introduction

There is a plethora of studies on the wave equation and it is difficult to even quote the most significant ones. We simply name [17] for a classical semigroup approach, since we will employ a semigroup approach as well, and [15], since it regards the wave equation from the port-Hamiltonian perspective, but of course plenty of other tools are available as well, e.g., [18]. The (classical) formulation of the wave equation on a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$ we are investigating in this article is the following second order partial differential equation

$$\begin{aligned}
\rho(\zeta)\tfrac{\partial^2}{\partial t^2} w(t,\zeta) &= \operatorname{div} T(\zeta)\nabla w(t,\zeta) - a(\zeta)w(t,\zeta) - b(\zeta)\tfrac{\partial}{\partial t} w(t,\zeta), && t \geq 0, \zeta \in \Omega, \\
w(0,\zeta) &= w_0(\zeta), && \zeta \in \Omega, \quad (1)\\
\tfrac{\partial}{\partial t} w(0,\zeta) &= w_1(\zeta), && \zeta \in \Omega.
\end{aligned}$$

For the initial discussion we will exclude the terms a and b, as we will take care of them with a perturbation argument later on. The coefficients T and ρ are the Young modulus and the material density, respectively. In the standard port-Hamiltonian approach, cf. [15], one introduces the new state variable $\left(\begin{smallmatrix} \nabla w \\ \rho w_t \end{smallmatrix}\right)$, which yields the following representation of the wave equation

$$\frac{\partial}{\partial t}\begin{pmatrix} \nabla w \\ \rho w_t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \nabla \\ \operatorname{div} & 0 \end{pmatrix}}_{=:\mathcal{J}_1} \underbrace{\begin{pmatrix} T & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix}}_{=:\mathcal{Q}_1} \begin{pmatrix} \nabla w \\ \rho w_t \end{pmatrix}.$$

This is the so-called *Dirac representation*, which is comprised of the formally skew-adjoint $\mathcal{J}_1$ (representing the underlying Dirac structure/subspace) and the bounded, positive and self-adjoint $\mathcal{Q}_1$ (representing the Lagrangian structure/subspace). However in [5] an alternative so-called *Lagrangian representation* was proposed, which

Date: May 14, 2025.
2020 *Mathematics Subject Classification.* 35L05, 35B35, 46N20, 47B02, 47B44.
Key words and phrases. Wave equation with split boundary, boundary triples, semi-uniform stability, rough coefficients, semigroups, dissipative operators, spectral theory.

uses $\left(\begin{smallmatrix} w \\ \rho w_t \end{smallmatrix}\right)$ as state variable. The corresponding system is

$$\frac{\partial}{\partial t}\begin{pmatrix} w \\ \rho w_t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}}_{=:\mathcal{J}_2}\underbrace{\begin{pmatrix} -\operatorname{div} T\nabla & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix}}_{=:\mathcal{Q}_2}\begin{pmatrix} w \\ \rho w_t \end{pmatrix}. \tag{2}$$

Similar to the Dirac representation, $\mathcal{J}_2$ is skew-adjoint and $\mathcal{Q}_2$ is formally self-adjoint and positive. In contrast to the Dirac-representation however, $\mathcal{J}_2$ is bounded and $\mathcal{Q}_2$ is unbounded. This formulation also corresponds to the standard method to turn the wave equation into a first order problem, cf. [17, Sec. 7.4].
Naturally, employing terms like (un)bounded and self-adjoint warrants specifying the state spaces. For the Dirac representation one chooses $\mathrm{L}^2(\Omega)^d \times \mathrm{L}^2(\Omega)$ and for the Lagrange representation it comes naturally to choose $\mathrm{H}^1(\Omega) \times \mathrm{L}^2(\Omega)$. In order to analyze well-posedness for the Dirac representation it suffices to analyze $\mathcal{J}_1$ as $\mathcal{Q}_1$ can be incorporated into the inner product and thus does not play a role, cf. [14, Lem. 7.2.3]. We want to mimic this approach for the Lagrange representation, but since $\mathcal{Q}_2$ is unbounded, it does not immediately provide an equivalent inner product on the state space, i.e., we want to regard the following almost inner product[1]

$$\begin{aligned}\langle x, y\rangle_{\mathcal{Q}_2} \coloneqq \langle \mathcal{Q}_2 x, y\rangle = \left\langle \begin{pmatrix} -\operatorname{div} T\nabla & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\rangle \\ = \langle \tfrac{1}{\rho}x_2, y_2\rangle - \langle \operatorname{div} T\nabla x_1, y_1\rangle.\end{aligned}$$

Applying integration by parts yields

$$\langle x, y\rangle_{\mathcal{Q}_2} = \langle \tfrac{1}{\rho}x_2, y_2\rangle + \langle T\nabla x_1, \nabla y_1\rangle - \big\langle \nu \cdot T\nabla x_1\big|_{\partial\Omega}, y_1\big|_{\partial\Omega}\big\rangle,$$

where ν denotes the unit normal vector on $\partial\Omega$. It is worth pointing out that this expression is only an inner product on the state space $\mathrm{H}^1(\Omega) \times \mathrm{L}^2(\Omega)$ if we additionally impose a boundary condition such as $\nu \cdot T\nabla x_1\big|_{\partial\Omega} = -k_1 x_1\big|_{\partial\Omega}$, where k_1 is a positive semi-definite operator[2] on $\mathrm{L}^2(\partial\Omega)$. In that case we obtain

$$\langle x, y\rangle_{\mathcal{Q}_2} = \langle \tfrac{1}{\rho}x_2, y_2\rangle + \langle T\nabla x_1, \nabla y_1\rangle + \big\langle k_1 x_1\big|_{\partial\Omega}, y_1\big|_{\partial\Omega}\big\rangle$$

and we can appeal to the Friedrichs/Poincaré inequality. Alternatively, the boundary condition $x_1\big|_{\partial\Omega} = 0 = y_1\big|_{\partial\Omega}$ would eliminate the boundary parts completely. In either case, the above expression induces an equivalent inner product on the state space $\mathrm{H}^1(\Omega) \times \mathrm{L}^2(\Omega)$. Note that the energy of a state $x \in \mathrm{H}^1(\Omega) \times \mathrm{L}^2(\Omega)$ is given by

$$\mathrm{E}(x) \coloneqq \langle \tfrac{1}{\rho}x_2, x_2\rangle + \langle T\nabla x_1, \nabla x_1\rangle.$$

Therefore the inner product $\langle \cdot, \cdot\rangle_{\mathcal{Q}_2}$ is composed of an energy part and a boundary part. We will construct a boundary triple for the operator in the abstract Cauchy equation (2), which will enable us to parameterize all dissipative boundary conditions, i.e., boundary conditions where the solution does not grow. Loosely speaking we will come to the conclusion that boundary conditions of the form

$$k_1 x_1\big|_{\partial\Omega} + \nu \cdot T\nabla x_1\big|_{\partial\Omega} + k_2 \tfrac{1}{\rho}x_2\big|_{\partial\Omega} = 0$$

or in terms of w

$$k_1 w\big|_{\partial\Omega} + \nu \cdot T\nabla w\big|_{\partial\Omega} + k_2 \partial_t w\big|_{\partial\Omega} = 0,$$

where k_1 is the positive semi-definite operator from before and k_2 is another positive semi-definite operator on $\mathrm{L}^2(\partial\Omega)$, will be well-posed. We point out that in contrast to the Dirac representation, cf. [15], we can formulate boundary conditions that

[1] We are a bit sloppy here in not specifying the regularity of the state variables.

[2] We are in particular interested in multiplication operators that may vanish on parts of the boundary.

involve the displacement w itself (additionally to the velocity $\partial_t w$ and the normal stress $\nu \cdot T\nabla w$).
Additionally, since k_1 and k_2 are allowed to be semi-definite we can split the boundary into five (possibly empty) parts and allow the following boundary conditions:

$$\begin{aligned} w &= 0 \quad \text{on } \Gamma_0 \\ \nu \cdot T\nabla w &= 0 \quad \text{on } \Gamma_1 \\ k_1 w + \nu \cdot T\nabla w &= 0 \quad \text{on } \Gamma_2 \\ \nu \cdot T\nabla w + k_2\partial_t w &= 0 \quad \text{on } \Gamma_3 \\ k_1 w + \nu \cdot T\nabla w + k_2\partial_t w &= 0 \quad \text{on } \Gamma_4 \end{aligned} \tag{3}$$

where we assume k_1 to be nonzero almost everywhere on $\Gamma_2 \cup \Gamma_4$ and k_2 to be nonzero almost everywhere on $\Gamma_3 \cup \Gamma_4$, i.e., the vanishing parts are being accounted for by Γ_0 and Γ_3 or by Γ_0, Γ_1 and Γ_2 respectively.
Note that $\Gamma_0 = \emptyset$ and $\Gamma_2 \cup \Gamma_4 = \emptyset$ ($k_1 = 0$) are allowed, but not simultaneously, i.e., we require

$$k_1 \neq 0 \quad \text{or} \quad \Gamma_0 \neq \emptyset.$$

This is necessary to make sure that $\langle \cdot, \cdot \rangle_{\mathcal{Q}_2}$ is an inner product.
In this work, boundary triples are used to show the well-posedness of the boundary conditions (3). In a related context, boundary triples have also proven useful in [11], where they are employed to investigate boundary conditions ensuring a Lagrangian subspace.
Following up on well-posedness, we will investigate stability of solutions. The dissipative relation on the boundary can be viewed as a boundary feedback or damping. Damped wave equations have been studied by multiple authors, in particular Zuazua (e.g., [25]), but more recently (even for the delay case) by Pignotti et. al. (e.g., [1, 16]). We point out, that opposed to our approach in this article, stability results for the wave equation usually cover the case, where the damping happens in the interior and geometric conditions (such as the Geometric Control Condition) are a necessary cost to pay.
We will pursue an alternative strategy, following [13] in our approach, settling for a weaker notion than exponential stability, by the name of "semi-uniform stability", cf. Section 4. Slightly stronger assumptions on the coefficients T and ρ in conjunction with dissipativity will prove enough to show at least semi-uniform stability of solutions to Equation (1). In particular, we have to show that there is no spectrum on the imaginary axis. A similar approach has been applied to Maxwell's equations in [21].

Assumptions. To turn this outline into rigorous mathematics, we now state the basic assumptions that will be used throughout the remainder of the article, unless explicitly stated otherwise.

(A1) Let Ω be a bounded and connected Lipschitz domain in $\mathbb{R}^d$.
(A2) Let $T \in \mathcal{L}_b(\mathrm{L}^2(\Omega; \mathbb{C}^d))$ with $c^{-1}\mathrm{I} < T < c\mathrm{I}$ for some $c > 0$.
For stability additionally: $T \in \mathrm{L}^\infty(\Omega; \mathbb{C}^{d\times d})$ and Lipschitz continuous.[3]
(A3) Let $\rho \in \mathcal{L}_b(\mathrm{L}^2(\Omega; \mathbb{C}))$ with $c^{-1}\mathrm{I} < \rho < c\mathrm{I}$ for some $c > 0$.
For stability additionally: $\rho \in \mathrm{L}^\infty(\Omega; \mathbb{C})$ and Lipschitz continuous.[3]
(A4) Let $w_0 \in \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$ and $w_1 \in \mathrm{L}^2(\Omega)$.[4]

[3]We identify the L^∞ function with the induced multiplication operator.
[4]Here $\mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega) = \{v \in \mathrm{H}^1(\Omega) \,|\, v|_{\Gamma_0} = 0\}$. We will come back to this space in Section 3.

(A5) The boundary is split into five open disjoint and possible empty parts $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \subseteq \partial\Omega$ that satisfy

$$\overline{\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4} = \partial\Omega \quad \text{and} \quad \sum_{i=0}^{4} \mu(\Gamma_i) = \mu(\partial\Omega),$$

where μ is the surface measure of $\partial\Omega$. These conditions ensure that the Γ's cover $\partial\Omega$ in the sense that the uncovered parts are negligible.

(A6) We assume $k_1, k_2 \in \mathrm{L}^\infty(\partial\Omega \setminus \overline{\Gamma_0}; \mathbb{R})$ such that $k_1, k_2 \geq 0$ and $k_1 \neq 0$ or $\Gamma_0 \neq \emptyset$. For stability additionally: $\exists \Gamma \subseteq \partial\Omega \setminus \overline{\Gamma_0}$ non-empty open such that $\operatorname{supp} k_2 \supseteq \Gamma$.[5]

(A7) Let $a, b \in \mathrm{L}^\infty(\Omega)$.[6]

Remark 1.1. Assumption (A6) is not as general as possible, because otherwise the formulation would be a bit cumbersome. We could relax the conditions to: $k_1, k_2 \in \mathcal{L}_\mathrm{b}(\mathrm{L}^2(\partial\Omega \setminus \overline{\Gamma_0}))$ positive semi-definite such that $1 \notin \ker k_1$ or $\Gamma_0 \neq \emptyset$. For stability we additionally ask for

$$k_2 f = 0 \quad \Longrightarrow \quad \exists \Gamma \subseteq \partial\Omega \setminus \overline{\Gamma_0} \text{ open } : f\big|_\Gamma = 0. \tag{4}$$

The condition $1 \notin \ker k_1$ is necessary for Friedrichs/Poincaré's inequality and (4) is necessary to apply the unique continuation principle.

In the following we first present a short preliminary section, which serves as a reminder for well-established concepts such as trace operators and boundary triples, which we will employ to prove our main results about well-posedness (Section 3) and stability (Section 4). We will end our article with a conclusion and a short appendix on the applicability of a unique continuation theorem in Section 4 and an additional result regarding regularity in the case of the trace maps taking values in $\mathrm{L}^2(\partial\Omega)$.

2. Preliminaries

This section is largely a short recapitulation of established theory that we present both for the convenience of the reader as well as to establish notation, that we are going to use in consecutive sections.

2.1. **Sobolev Spaces.** We clarify/introduce some notation first: We let $\mathrm{H}^1(\Omega)$ be the space of all L^2-functions with distributional derivative also in $\mathrm{L}^2(\Omega)$ together with the norm $\|\cdot\|_{\mathrm{H}^1(\Omega)} := \sqrt{\|\cdot\|^2_{\mathrm{L}^2(\Omega)} + \|\nabla\cdot\|^2_{\mathrm{L}^2(\Omega)}}$. In other words, $\mathrm{H}^1(\Omega) = \big(\operatorname{dom}(\nabla), \|\cdot\|_{\operatorname{dom}(\nabla)}\big)$, where ∇ is the weak gradient on $\mathrm{L}^2(\Omega)$. Similarly, the corresponding space for the divergence operator $\operatorname{div} f := \sum_{i=1}^d \partial_i f_i$ is

$$\mathrm{H}(\operatorname{div}, \Omega) := \{ f \in \mathrm{L}^2(\Omega) \,|\, \operatorname{div} f \in \mathrm{L}^2(\Omega) \text{ (in the distributional sense)} \}.$$

Note that $\mathrm{C}^\infty_\mathrm{c}(\overline{\Omega}) := \big\{ f\big|_{\overline{\Omega}} \,\big|\, f \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^d) \big\}$ is dense in $\mathrm{H}(\operatorname{div}, \Omega)$, see, e.g., [7, Ch. IX Part A Sec. 2 Thm. 1] or [20, Thm. 3.18].

2.2. **Trace Operators.** The following can be found in more detail in [15, Appendix A]. For $f \in \mathrm{C}^\infty_\mathrm{c}(\overline{\Omega})$ we can define the map

$$\tilde{\gamma}_0 \colon \left\{ \begin{array}{ccc} \mathrm{C}^\infty_\mathrm{c}(\overline{\Omega}) & \to & \mathrm{L}^2(\partial\Omega), \\ f & \mapsto & f\big|_{\partial\Omega}. \end{array} \right.$$

[5]This just means that the damping acts on an open set.

[6]More general assumptions are possible, cf. sections 3 and 4.

By continuous extension we can extend $\tilde{\gamma}_0$ to a continuous map $\gamma_0\colon \mathrm{H}^1(\Omega) \to \mathrm{L}^2(\partial\Omega)$, the *Dirichlet trace operator*. We define its image as

$$\mathrm{H}^{\frac{1}{2}}(\partial\Omega) \coloneqq \operatorname{ran}\gamma_0 \quad \text{equipped with} \quad \|\phi\|_{\mathrm{H}^{\frac{1}{2}}(\partial\Omega)} \coloneqq \inf\{\|g\|_{\mathrm{H}^1(\Omega)} \mid \gamma_0 g = \phi\}.$$

This space is even a Hilbert space as it can be represented by the quotient space $\mathrm{H}^1(\Omega)/\ker\gamma_0$. Note that γ_0 is a continuous map from $\mathrm{H}^1(\Omega)$ to $\mathrm{H}^{\frac{1}{2}}(\partial\Omega)$ and $\mathrm{H}^{\frac{1}{2}}(\partial\Omega)$ is continuously embedded in $\mathrm{L}^2(\partial\Omega)$. Let $\Gamma_0, \tilde{\Gamma} \subseteq \partial\Omega$ be a (measurable) partition of $\partial\Omega$ (up to sets of measure zero). We can restrict $\gamma_0 f$ to Γ_0, which gives that $\gamma_0\big|_{\Gamma_0}\colon f \mapsto \gamma_0 f\big|_{\Gamma_0}$ is continuous from $\mathrm{H}^1(\Omega)$ to $\mathrm{L}^2(\Gamma_0)$ also. Hence,

$$\mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega) \coloneqq \ker\gamma_0\big|_{\Gamma_0} = \{f \in \mathrm{H}^1(\Omega) \mid \gamma_0 f\big|_{\Gamma_0} = 0\}$$

is closed and therefore a Hilbert space equipped with $\|\cdot\|_{\mathrm{H}^1(\Omega)}$. We define the corresponding trace space

$$\mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma}) \coloneqq \gamma_0 \mathring{\mathrm{H}}^1_{\Gamma_0}(\partial\Omega)$$

and endow it with $\|\cdot\|_{\mathrm{H}^{\frac{1}{2}}(\partial\Omega)}$. As the functions in $\mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma})$ are 0 on Γ_0 we usually regard this space as continuously embedded in $\mathrm{L}^2(\tilde{\Gamma})$. We define the dual space of $\mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma})$ with pivot space $\mathrm{L}^2(\tilde{\Gamma})$ as

$$\mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma}) \coloneqq \big(\mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma})\big)',$$

i.e., $\mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma}) \subseteq \mathrm{L}^2(\tilde{\Gamma}) \subseteq \mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma})$ forms a Gelfand triple. The pairing between $\mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma})$ and $\mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma})$ is an extension of the inner product of $\mathrm{L}^2(\tilde{\Gamma})$:

$$\langle f, g\rangle_{\mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma}), \mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma})} \coloneqq \lim_{n\to\infty} \langle f_n, g\rangle_{\mathrm{L}^2(\tilde{\Gamma})},$$

where $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathrm{L}^2(\tilde{\Gamma})$ converging to $f \in \mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma})$ (w.r.t. $\|\cdot\|_{\mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma})}$) and $g \in \mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma})$. Moreover, we define $\langle g, f\rangle_{\mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma}), \mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma})}$ as the complex conjugate of $\langle f, g\rangle_{\mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma}), \mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma})}$ and we will use the short notation

$$\langle f, g\rangle_{\mp\frac{1}{2}} \coloneqq \langle f, g\rangle_{\mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma}), \mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma})} \quad \text{and} \quad \langle g, f\rangle_{\pm\frac{1}{2}} \coloneqq \langle g, f\rangle_{\mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma}), \mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma})}.$$

For $f \in \mathrm{C}_\mathrm{c}^\infty(\overline{\Omega})^d$ we can define the map

$$\tilde{\gamma}_\nu\colon \left\{ \begin{array}{rcl} \mathrm{C}_\mathrm{c}^\infty(\overline{\Omega})^d & \to & \mathrm{L}^2(\tilde{\Gamma}), \\ f & \mapsto & \nu \cdot f\big|_{\tilde{\Gamma}}. \end{array} \right.$$

By continuous extension we can extend $\tilde{\gamma}_\nu$ to $\gamma_\nu\colon \mathrm{H}(\operatorname{div}, \Omega) \to \mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma})$, the *normal trace operator.*

Theorem 2.1 ([15, Theorem A.8])**.** *The* normal trace operator $\gamma_\nu\colon \mathrm{H}(\operatorname{div}, \Omega) \to \mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma})$ *is a bounded, linear and surjective operator.*

Both trace operators together give rise to the following integration by parts rule:

Theorem 2.2 (Integration by parts)**.** *For any* $F \in \mathrm{H}(\operatorname{div}, \Omega)$ *and* $g \in \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$ *there holds:*

$$\langle \operatorname{div} F, g\rangle_{\mathrm{L}^2(\Omega)} = \langle F, \nabla g\rangle_{\mathrm{L}^2(\Omega)^d} + \langle \gamma_\nu F, \gamma_0 g\rangle_{\mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma}), \mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma})}$$

Proof. For $F \in \mathrm{C}^\infty(\overline{\Omega})^d$ we have for $g \in \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$:

$$\int_\Omega \langle \operatorname{div} F, g\rangle_\mathbb{C} \,\mathrm{d}\lambda = \int_\Omega \langle F, \nabla g\rangle_{\mathbb{C}^d} \,\mathrm{d}\lambda + \int_{\tilde{\Gamma}} (\nu \cdot F) g \,\mathrm{d}\mu$$

Density of $\mathrm{C}_\mathrm{c}^\infty(\overline{\Omega})^d$ in $\mathrm{H}(\operatorname{div}, \Omega)$ implies the claim. ❑

The Friedrichs/Poincaré inequality (Theorem C.1) allows $\mathrm{H}^1(\Omega)$ to be equipped with an equivalent inner product:

$$\langle f, g\rangle_{\mathrm{H}^1(\Omega)} \coloneqq \langle \nabla f, \nabla g\rangle_{\mathrm{L}^2(\Omega)} + \langle k_1 \gamma_0 f, \gamma_0 g\rangle_{\mathrm{L}^2(\tilde{\Gamma})}. \tag{5}$$

In particular we are interested in $\mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$, which is a subset of $\mathrm{H}^1(\Omega)$. Note that (5) gives still an equivalent norm on $\mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$ if $k_1 = 0$, under the condition that $\Gamma_0 \neq \emptyset$. Hence, we either need

$$k_1 \neq 0 \quad \text{or} \quad \Gamma_0 \neq \emptyset.$$

2.3. **Dissipative operators.** We remind the reader of a few commonly known facts about dissipative operators:

Definition 2.3. A linear operator $A\colon H \supseteq \operatorname{dom} A \to H$ on a Hilbert space H is called *dissipative*, if $\operatorname{Re}\langle Ax, x\rangle_H \leq 0$ for all $x \in \operatorname{dom} A$. A is called *maximally dissipative* if there is no proper dissipative extension of A.

Here are a few characterizations of maximally dissipative operators:

Proposition 2.4. *Let A be a densely defined linear operator on a Hilbert space H. The following are equivalent:*
- (i) *A is maximally dissipative.*
- (ii) *A is dissipative and* $\operatorname{ran}(\lambda I - A)$ *is onto for some (and hence all) $\lambda > 0$.*
- (iii) *A is closed and both A and A^* are dissipative.*

Proofs of these statements as well as the next theorem are contained in [9, Sec. II.3b]. We will rely on the following well-known theorem:

Theorem 2.5 (Lumer–Philips)**.** *Let $A\colon H \supseteq \operatorname{dom} A \to H$ be a densely defined linear operator on a Hilbert space H. Then A is infinitesimal generator of a contraction semigroup if and only if A is maximally dissipative.*

2.4. **Helmholtz decomposition.** The classical Helmholtz decomposition allows the splitting of a vector field into a gradient field and a divergence-free vector field. We will also use a slight modification of the classical Helmholtz decomposition and introduce

$$\mathring{\mathrm{H}}^1(\Omega) \coloneqq \mathring{\mathrm{H}}^1_{\partial\Omega}(\Omega) = \{f \in \mathrm{H}^1(\Omega) \mid \gamma_0 f = 0\}$$
$$\mathrm{H}(\operatorname{div} 0, \Omega) \coloneqq \{v \in \mathrm{H}(\operatorname{div}, \Omega) \mid \operatorname{div} v = 0\} = \ker \operatorname{div}.$$

With this preparation we can show the following modification of the classical result:

Theorem 2.6 (Helmholtz decomposition)**.** *Let $T \in \mathcal{L}_{\mathrm{b}}(\mathrm{L}^2(\Omega))$ be boundedly invertible. Then*

$$\mathrm{L}^2(\Omega) = T\nabla\mathring{\mathrm{H}}^1(\Omega) \oplus_{T^{-1}} \mathrm{H}(\operatorname{div} 0, \Omega),$$

where $\oplus_{T^{-1}}$ denotes the orthogonal sum w.r.t. the inner product $\langle f, g\rangle_{T^{-1}} \coloneqq \langle T^{-1} f, g\rangle$.

Proof. The standard Helmholtz decomposition with $T = \mathrm{id}$ is simply an application of the orthogonal decomposition of $\mathrm{L}^2(\Omega)$ into $\overline{\operatorname{ran} L}$ and $\ker L^*$, where L is the densely defined linear operator $\mathring{\nabla}$, i.e., the gradient with Dirichlet boundary conditions (∇ restricted to $\mathring{\mathrm{H}}^1(\Omega) \subseteq \mathrm{L}^2(\Omega)$).

Hence, we only have to verify closedness of $\operatorname{ran} L$. Let $\{u_n\}_n \subseteq \operatorname{ran} L$ with $u_n = L v_n$ for some $v_n \in \mathrm{L}^2(\Omega)$ converge to some $u \in \mathrm{L}^2(\Omega)$. Convergence of $(u_n)_n$ implies boundedness of $(L v_n)_n$ in $\mathrm{L}^2(\Omega)$. Since $\|\nabla\cdot\|_{\mathrm{L}^2(\Omega)}$ and $\|\cdot\|_{\mathrm{H}^1(\Omega)}$ are equivalent on $\mathring{\mathrm{H}}^1(\Omega)$ by virtue of the standard Poincaré inequality, this implies boundedness of $(v_n)_n$ as a sequence in $\mathrm{H}^1(\Omega)$. By Rellich's theorem, there exists a strongly

convergent subsequence of $(v_n)_n$ in $\mathrm{L}^2(\Omega)$. Appealing to the closedness of L, we obtain that $u = \lim_{n\to\infty} u_n = \lim_{n\to\infty} Lv_n = L \lim_{n\to\infty} v_n$, hence $u \in \operatorname{ran} L$.
The T in front of the first orthogonal space and the T^{-1} attached in the first component of the inner product corresponding to $\oplus_{T^{-1}}$ cancel out and give the standard Helmholtz decomposition. ❑

2.5. **Boundary triples.** Boundary triples were originally developed for symmetric operators see [12, 4], but they can be equivalently defined in the skew-symmetric case, see [23] and [19, Ch. 2.4]. This is a simple consequence of multiplying the equations by the imaginary unit i. We will present the version from [19] that allows dual pairs as the boundary spaces instead of a single (identified) Hilbert space. There is also the notion of m-boundary tuples [8], which additionally requires a pivot space for the dual pair.

Definition 2.7. Let A_0 be a densely defined, skew-symmetric and closed operator on a Hilbert space H. A *boundary triple* for A_0^* is a triple $\big((\mathcal{B}_+, \mathcal{B}_-), B_1, B_2\big)$ consisting of a complete dual pair[7] $(\mathcal{B}_+, \mathcal{B}_-)$ and two operators $B_1\colon \operatorname{dom}(A_0^*) \to \mathcal{B}_+$ and $B_2\colon \operatorname{dom}(A_0^*) \to \mathcal{B}_-$ such that

(i) The map
$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} : \begin{cases} \operatorname{dom}(A_0^*) & \to \quad \mathcal{B}_+ \times \mathcal{B}_-, \\ \qquad x & \mapsto \quad \begin{pmatrix} B_1 x \\ B_2 x \end{pmatrix} \end{cases}$$
is onto.

(ii) The following abstract Green's identity holds for all $x, y \in \operatorname{dom}(A_0^*)$:
$$\langle A_0^* x, y\rangle_X + \langle x, A_0^* y\rangle_X = \langle B_1 x, B_2 y\rangle_{\mathcal{B}_+,\mathcal{B}_-} + \langle B_2 x, B_1 y\rangle_{\mathcal{B}_-,\mathcal{B}_+}.$$

The abstract Green's identity (ii) formalizes an integration by parts formula, which is the reason for the designation boundary triple.
A boundary triple enables us to parameterize all boundary conditions such that A_0^* restricted to all elements of $\operatorname{dom} A_0^*$, that satisfy the boundary condition, is maximally dissipative. In our situation we will obtain a boundary triple that involves the trace spaces $\mathring{\mathrm{H}}^{\frac{1}{2}}(\Gamma)$ and $\mathrm{H}^{-\frac{1}{2}}(\Gamma)$, but we would rather formulate boundary conditions in the pivot space $\mathrm{L}^2(\Gamma)$. Verifying dissipativity is usually straightforward, but the maximality can be tricky. In the next section, we will arrive at a situation in which $((\mathcal{B}_-, \mathcal{B}_+), B_1, B_2)$ is a boundary triple[8], where the duality of $(\mathcal{B}_+, \mathcal{B}_-)$ is induced by a pivot space $\mathcal{B}_0$. Moreover, $\mathcal{B}_+ \subseteq \mathcal{B}_0 \subseteq \mathcal{B}_-$ forms a Gelfand triple. We want to formulate Robin type boundary conditions of the form
$$B_1 x + \Theta B_2 x = 0,$$
where $\Theta \in \mathcal{L}_\mathrm{b}(\mathcal{B}_0)$. So strictly speaking we have to take the embedding mappings $j_+\colon \mathcal{B}_+ \to \mathcal{B}_0$ and $j_-\colon \mathcal{B}_0 \to \mathcal{B}_-$ into account, i.e.,
$$j_-^{-1} B_1 x + \Theta j_+ B_2 x = 0$$
or equivalently
$$B_1 x = -j_- \Theta j_+ B_2 x.$$
The corresponding operator is
$$\begin{aligned} A_\Theta &\coloneqq A_0^*\big|_{\operatorname{dom} A_\Theta}, \\ \operatorname{dom} A_\Theta &\coloneqq \{x \in \operatorname{dom} A_0^* \mid B_1 x = -j_- \Theta j_+ B_2 x\}. \end{aligned} \tag{6}$$

---

[7]*Complete dual pair* simply means that that the spaces are dual to each other, but we do not identify them by means of the Riesz isomorphism in the case of a Hilbert space.

[8]The order of $(\mathcal{B}_-, \mathcal{B}_+)$ is swapped on purpose, because later we will use $(\mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma}), \mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma}))$.

We denote the induced operator between $\mathcal{B}_+$ and $\mathcal{B}_-$ by $\hat{\Theta}$, i.e.,

$$\hat{\Theta} := -j_-\Theta j_+.$$

By the theory of boundary triples A_Θ is maximally dissipative if and only if $\hat{\Theta}$ is maximally dissipative, see, e.g., [19, Prop. 2.4.10] or [4, Cor. 2.1.4.].[9]

Proposition 2.8. *Let $(\mathcal{B}_+, \mathcal{B}_0, \mathcal{B}_-)$ be a Gelfand triple, $((\mathcal{B}_-, \mathcal{B}_+), B_1, B_2)$ be a boundary triple for A_0^* and $\Theta \in \mathcal{L}_\mathrm{b}(\mathcal{B}_0)$. Then A_Θ is maximally dissipative, if Θ is positive (semi-definite), i.e., $\langle \Theta h, h\rangle_{\mathcal{B}_0} \geq 0$ for all $h \in \mathcal{B}_0$.*

Proof. Note that A_Θ is maximally dissipative, if $\hat{\Theta} := -j_-\Theta j_+$ is maximally dissipative. Hence, it is sufficient to show (the stronger assertion) that $\hat{\Theta}$ is self-adjoint and negative.
Since j_+ and j_- are continuous embeddings we have $\hat{\Theta} \in \mathcal{L}_\mathrm{b}(\mathcal{B}_+, \mathcal{B}_-)$. Moreover, $j_+^* = j_-$ and $j_-^* = j_+$.[10] This implies

$$\operatorname{Re}\langle \hat{\Theta} x, x\rangle_{\mathcal{B}_-,\mathcal{B}_+} = -\operatorname{Re}\langle \Theta j_+ x, j_+ x\rangle_{\mathcal{B}_0} \leq 0.$$

Furthermore we have

$$\hat{\Theta}^* = -(j_-\Theta j_+)^* = -j_+^*\Theta^* j_-^* = -(j_-\Theta j_+)^* = \hat{\Theta},$$

which implies that $\hat{\Theta}$ is self-adjoint and therefore maximally dissipative. Hence, also $\hat{\Theta}^{-1}$ is maximally dissipative (in the sense of linear relations). Finally, we can apply [19, Prop. 2.4.10], which deduces the maximal dissipativity of A_Θ from the maximal dissipativity of $\hat{\Theta}^{-1}$ (Note that [19] uses a different notation. The operator A_Θ corresponds to $A_{\hat{\Theta}^{-1}}$ in the notation of [19]). ❑

3. Well-posedness

In [15] it was shown that there is a boundary triple associated to the wave equation in the Dirac representation. In this section we show that we can also associate a boundary triple to the wave equation in the Lagrange representation. This will allow us to parameterize all maximally dissipative boundary conditions.
Recall the decomposition of $\partial\Omega$ into Γ_0, Γ_1, Γ_2 Γ_3 and Γ_4. Since the case where Γ_0 is equal to $\partial\Omega$ simply corresponds to the wave equation with Dirichlet boundary, which is well-studied, we exclude that case. In particular, we will see, that the boundary conditions (3) is one of these maximally dissipative boundary conditions. For the following we only have to distinguish between Γ_0 and the rest. Hence, let us abbreviate $\tilde{\Gamma} := \partial\Omega \setminus \overline{\Gamma_0}$, which is essentially the same as $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

Reformulation. First we want to transfer the wave equation into the formalism for boundary triples. As state space we choose

$$X := \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega) \times \mathrm{L}^2(\Omega).$$

We use the equivalent norm for $\mathrm{H}^1(\Omega)$ given by the Poincaré inequality Theorem C.1 and define

$$\langle x, y\rangle_X := \left\langle \tfrac{1}{\rho} x_2, y_2 \right\rangle_{\mathrm{L}^2(\Omega)} + \langle T\nabla x_1, \nabla y_1\rangle_{\mathrm{L}^2(\Omega)} + \langle k_1\gamma_0 x_1, \gamma_0 y_1\rangle_{\mathrm{L}^2(\tilde{\Gamma})}.$$

Equivalence to the canonical inner product on X is assured by virtue of the Poincaré inequality and the assumptions on ρ, T and k_1. This is the energy inner product that is favored in port-Hamiltonian formulations plus the inner product on the boundary, which is necessary to prevent constant functions in the second argument to have vanishing norm. We arrive (informally) at the differential operator A describing the

[9] In the second reference boundary triples are introduced for the symmetric setting.

[10] We regard the adjoint w.r.t. the dual pair $(\mathcal{B}_+, \mathcal{B}_-)$, i.e., if $A\colon \mathcal{B}_+ \to H$, then $A^*\colon H \to \mathcal{B}_-$ and analogously if $\mathcal{B}_+$ is the codomain. It is the reverse way around for $\mathcal{B}_-$.

wave equation (1) by applying the classical substitutions $x_1 = \rho\partial_t w$ and $x_2 = w$ and obtain:

$$\partial_t \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix} \begin{pmatrix} -\operatorname{div} T\nabla & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix}}_{=A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -a & -b\frac{1}{\rho} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Formally we define:

$$A\colon X \supseteq \operatorname{dom}(A) \to X, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{\rho}x_2 \\ \operatorname{div} T\nabla x_1 \end{pmatrix}, \tag{7}$$

where

$$\operatorname{dom} A := \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X \;\middle|\; \tfrac{1}{\rho}x_2 \in \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega) \text{ and } T\nabla x_1 \in \mathrm{H}(\operatorname{div},\Omega) \right\}.$$

Lemma 3.1. *A is a closed operator.*

Proof. Let $(\binom{x_n}{y_n})_{n\in\mathbb{N}}$ be a sequence in $\operatorname{dom} A$ that converges to $\binom{x}{y} \in X$ w.r.t. $\|\cdot\|_X$ such that also $(A\binom{x_n}{y_n})_{n\in\mathbb{N}}$ converges to $\binom{v}{w} \in X$ w.r.t. $\|\cdot\|_X$. Note that $A\binom{x_n}{y_n} = \binom{\frac{1}{\rho}y_n}{\operatorname{div} T\nabla x_n}$ implies that $\frac{1}{\rho}y_n$ converges to v w.r.t. $\|\cdot\|_{\mathrm{H}^1(\Omega)}$ and therefore $v = \frac{1}{\rho}y$. Since x_n converges to x w.r.t. $\|\cdot\|_{\mathrm{H}^1(\Omega)}$ we conclude that $T\nabla x_n$ converges to $T\nabla x$ w.r.t. $\|\cdot\|_{\mathrm{L}^2(\Omega)}$. Hence, the closedness of div implies $w = \operatorname{div} T\nabla x$, which shows the closedness of A. ❑

A is what we have called A_0^* in Definition 2.7 for an (at this point still undefined) operator A_0.[11] As outlined in Section 2, we want to attach boundary conditions to A by identifying a suitable boundary triple and then define A_Θ. For our boundary operators we define:

$$B_1\colon \begin{cases} \operatorname{dom} A & \to \quad \mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma}), \\ \quad x & \mapsto \quad k_1\gamma_0 x_1 + \gamma_\nu T\nabla x_1, \end{cases} \quad \text{and } B_2\colon \begin{cases} \operatorname{dom} A & \to \quad \mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma}), \\ \quad x & \mapsto \quad \gamma_0\frac{1}{\rho}x_2. \end{cases} \tag{8}$$

The ultimate goal of this section is to show that $\big((\mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma}), \mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma})), B_1, B_2\big)$ is a boundary triple for A. From this, well-posedness of Equation (1) will easily follow. According to Definition 2.7 we have to verify:

(i) $B := \binom{B_1}{B_2}$ is onto.
(ii) A and B satisfy an abstract Green identity.

To verify surjectivity of B it is enough to show that the individual B_1 and B_2 are surjective, as they act on different components of $x = \binom{x_1}{x_2}$. We first need a very useful result about solutions of the (generalized) Dirichlet equation with boundary condition provided by B_1. The result is a slight modification of [22, Thm. 5.5]:

Proposition 3.2. *For any $g \in \mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma})$ there exists $w \in \{f \in \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega) \,|\, T\nabla f \in \mathrm{H}(\operatorname{div},\Omega)\}$ such that*

$$\begin{aligned} \operatorname{div} T\nabla w &= 0 \quad \text{in } \Omega, \\ \gamma_\nu T\nabla w + k_1\gamma_0 w &= g \quad \text{on } \tilde{\Gamma}. \end{aligned}$$

In particular, $w = \gamma_0^* g$ is given by the adjoint

$$\gamma_0^*\colon \mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma}) \to \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega) \quad \text{of} \quad \gamma_0\colon \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega) \to \mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma}),$$

where $\mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$ is equipped with

$$\langle f, g\rangle_{\nabla,\gamma_0} := \langle T\nabla f, \nabla g\rangle_{\mathrm{L}^2(\Omega)} + \langle k_1\gamma_0 f, \gamma_0 g\rangle_{\mathrm{L}^2(\tilde{\Gamma})}.$$

[11] In fact, we will forego the definition of A_0 entirely.

Proof. Note that $\gamma_0 \colon \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega) \to \mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma})$ is bounded. Thus, its adjoint is defined on all of $\mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma})$ and is bounded. Moreover, $\gamma_0^* g \in \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$ as this is the range of γ_0^*. By definition of the adjoint we have for $v \in \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$

$$\langle g, \gamma_0 v\rangle_{\mp\frac{1}{2}} = \langle \gamma_0^* g, v\rangle_{\nabla,\gamma_0} = \langle T\nabla\gamma_0^* g, \nabla v\rangle_{\mathrm{L}^2(\Omega)} + \langle k_1\gamma_0\gamma_0^* g, \gamma_0 v\rangle_{\mathrm{L}^2(\tilde{\Gamma})}.$$

If we choose $v \in \mathrm{C}_c^\infty(\Omega)$ we obtain

$$0 = \langle T\nabla\gamma_0^* g, \nabla v\rangle_{\mathrm{L}^2(\Omega)},$$

which implies $T\nabla\gamma_0^* g \in \mathrm{H}(\mathrm{div}, \Omega)$ and $\operatorname{div} T\nabla\gamma_0^* g = 0$. Hence, for arbitrary $v \in \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$ integration by parts gives

$$\begin{aligned}\langle g, \gamma_0 v\rangle_{\mp\frac{1}{2}} &= \langle \gamma_\nu T\nabla\gamma_0^* g, \gamma_0 v\rangle_{\mp\frac{1}{2}} + \langle k_1\gamma_0\gamma_0^* g, \gamma_0 v\rangle_{\mp\frac{1}{2}} \\ &= \langle \gamma_\nu T\nabla\gamma_0^* g + k_1\gamma_0\gamma_0^* g, \gamma_0 v\rangle_{\mp\frac{1}{2}},\end{aligned}$$

which in turn implies $w = \gamma_0^* g$, finishing the proof. ❑

Corollary 3.3. *B is surjective.*

Proof. Proposition 3.2 shows that B_1 is surjective and B_2 is surjective as a consequence of the surjectivity of the Dirichlet trace operator. Since B_1 and B_2 act on different components, we obtain the surjectivity of B. ❑

For the second step we observe that integration by parts for the Dirichlet and normal trace operators (Theorem 2.2) gives rise to an abstract Green identity:

Proposition 3.4 (Green identity)**.** *For all $x, y \in \operatorname{dom} A$ the following identity holds:*

$$\langle Ax, y\rangle + \langle x, Ay\rangle = \langle B_1 x, B_2 y\rangle + \langle B_2 x, B_1 y\rangle.$$

Proof. Let $x, y \in \operatorname{dom}(A)$. We can calculate:

$$\begin{aligned}&\langle Ax, y\rangle_X + \langle x, Ay\rangle_X \\ &\quad= \left\langle \begin{pmatrix} \frac{1}{\rho}x_2 \\ \operatorname{div} T\nabla x_1\end{pmatrix}, \begin{pmatrix} y_1 \\ y_2\end{pmatrix}\right\rangle_X + \left\langle \begin{pmatrix} x_1 \\ x_2\end{pmatrix}, \begin{pmatrix} \frac{1}{\rho}y_2 \\ \operatorname{div} T\nabla y_1\end{pmatrix}\right\rangle_X\end{aligned}$$

By the definition of the inner product in X we have

$$\begin{aligned}&= \langle \tfrac{1}{\rho}\operatorname{div} T\nabla x_1, y_2\rangle_{\mathrm{L}^2(\Omega)} + \langle T\nabla\tfrac{1}{\rho}x_2, \nabla y_1\rangle_{\mathrm{L}^2(\Omega)} + \langle k_1\gamma_0\tfrac{1}{\rho}x_2, \gamma_0 y_1\rangle_{\mathrm{L}^2(\tilde{\Gamma})} \\ &\quad+ \langle x_2, \tfrac{1}{\rho}\operatorname{div} T\nabla y_1\rangle_{\mathrm{L}^2(\Omega)} + \langle \nabla x_1, T\nabla\tfrac{1}{\rho}y_2\rangle_{\mathrm{L}^2(\Omega)} + \langle \gamma_0 x_1, k_1\gamma_0\tfrac{1}{\rho}y_2\rangle_{\mathrm{L}^2(\tilde{\Gamma})}\end{aligned}$$

Applying integration by parts on the $\operatorname{div} T\nabla$ operators gives (we suppress the index for the inner product in the following and just distinguish between inner products and dual pairings)

$$\begin{aligned}&= -\langle T\nabla x_1, \nabla\tfrac{1}{\rho}y_2\rangle + \langle \gamma_\nu T\nabla x_1, \gamma_0\tfrac{1}{\rho}y_2\rangle_{\mp\frac{1}{2}} + \langle T\nabla\tfrac{1}{\rho}x_2, \nabla y_1\rangle + \langle k_1\gamma_0\tfrac{1}{\rho}x_2, \gamma_0 y_1\rangle \\ &\quad- \langle \nabla\tfrac{1}{\rho}x_2, T\nabla y_1\rangle + \langle \gamma_0\tfrac{1}{\rho}x_2, \gamma_\nu T\nabla y_1\rangle_{\pm\frac{1}{2}} + \langle \nabla x_1, T\nabla\tfrac{1}{\rho}y_2\rangle + \langle \gamma_0 x_1, k_1\gamma_0\tfrac{1}{\rho}y_2\rangle\end{aligned}$$

This simplifies to

$$\begin{aligned}&= \langle \gamma_\nu T\nabla x_1, \gamma_0\tfrac{1}{\rho}y_2\rangle_{\mp\frac{1}{2}} + \langle k_1\gamma_0\tfrac{1}{\rho}x_2, \gamma_0 y_1\rangle \\ &\quad+ \langle \gamma_0\tfrac{1}{\rho}x_2, \gamma_\nu T\nabla y_1\rangle_{\pm\frac{1}{2}} + \langle \gamma_0 x_1, k_1\gamma_0\tfrac{1}{\rho}y_2\rangle \\ &= \langle k_1\gamma_0 x_1 + \gamma_\nu T\nabla x_1, \gamma_0\tfrac{1}{\rho}y_2\rangle_{\mp\frac{1}{2}} + \langle \gamma_0\tfrac{1}{\rho}x_2, k_1\gamma_0 y_1 + \gamma_\nu T\nabla y_1\rangle_{\pm\frac{1}{2}} \\ &= \langle B_1 x, B_2 y\rangle_{\mp\frac{1}{2}} + \langle B_2 x, B_1 y\rangle_{\pm\frac{1}{2}}\end{aligned}$$

❑

To avoid having to define A_0 (from Definition 2.7) itself and having to verify that $A_0^* = A$, we will simply prove $A^* \subseteq -A$. This way, $A_0 \coloneqq -A^*$ has all necessary properties of A_0, i.e., skew-symmetry, dense domain and closedness.[12]

Proposition 3.5. *$A^* \subseteq -A$ and A is densely defined*

Note that the following proof must be formulated in the language of linear relations, since we do not know a priori whether the adjoint of A is single-valued—which is equivalent to A being densely defined. This has the useful side effect that the density of A's domain follows automatically.

Proof. Let $(y, z) \in A^*$ (where $y = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right) \in X$ and $z = \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right) \in X$). Then we have for $x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \operatorname{dom} A$

$$\langle Ax, y\rangle_X = \langle x, z\rangle_X$$

or equivalently

$$\begin{aligned}\langle \tfrac{1}{\rho} \operatorname{div} T\nabla x_1, y_2\rangle_{\mathrm{L}^2(\Omega)} &+ \langle T\nabla \tfrac{1}{\rho} x_2, \nabla y_1\rangle_{\mathrm{L}^2(\Omega)} + \langle k_1\gamma_0 \tfrac{1}{\rho} x_2, \gamma_0 y_1\rangle_{\mathrm{L}^2(\tilde\Gamma)} \\ &= \langle \tfrac{1}{\rho} x_2, z_2\rangle_{\mathrm{L}^2(\Omega)} + \langle T\nabla x_1, \nabla z_1\rangle_{\mathrm{L}^2(\Omega)} + \langle k_1\gamma_0 x_1, \gamma_0 z_1\rangle_{\mathrm{L}^2(\tilde\Gamma)}. \quad (9)\end{aligned}$$

Let $x_2 \in \mathrm{C}_\mathrm{c}^\infty(\Omega)$. Then we choose $x = \left(\begin{smallmatrix} 0 \\ \rho x_2 \end{smallmatrix}\right)$, which is in $\operatorname{dom} A$, and obtain

$$\langle \nabla x_2, T\nabla y_1\rangle_{\mathrm{L}^2(\Omega)} = \langle x_2, z_2\rangle_{\mathrm{L}^2(\Omega)}.$$

This holds true for all $x_2 \in \mathrm{C}_\mathrm{c}^\infty(\Omega)$, hence $T\nabla y_1 \in \mathrm{H}(\operatorname{div}, \Omega)$ and $z_2 = -\operatorname{div} T\nabla y_1$. For arbitrary $g \in \mathrm{H}^{-\frac{1}{2}}(\tilde\Gamma)$ we let $x_1 = \gamma_0^* g$. Appealing to Proposition 3.2 we conclude $B_1\left(\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right) = g$ and $\operatorname{div} T\nabla x_1 = 0$. This assures $x = \left(\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right) \in \operatorname{dom} A$, which allows us to calculate

$$\begin{aligned}0 = \langle \tfrac{1}{\rho} \underbrace{\operatorname{div} T\nabla x_1}_{=0}, y_2\rangle_{\mathrm{L}^2(\Omega)} &\overset{(9)}{=} \langle T\nabla x_1, \nabla z_1\rangle_{\mathrm{L}^2(\Omega)} + \langle k_1\gamma_0 x_1, \gamma_0 z_1\rangle_{\mathrm{L}^2(\tilde\Gamma)} \\ &= 0 + \langle \gamma_\nu T\nabla x_1, \gamma_0 z_1\rangle_{\mp\frac{1}{2}} + \langle k_1\gamma_0 x_1, \gamma_0 z_1\rangle_{\mathrm{L}^2(\tilde\Gamma)} \\ &= \langle g, \gamma_0 z_1\rangle_{\mp\frac{1}{2}}.\end{aligned}$$

Since $g \in \mathrm{H}^{-\frac{1}{2}}(\tilde\Gamma)$ was arbitrary, we conclude $\gamma_0 z_1 = 0$. Appealing to the Helmholtz decomposition $\mathrm{L}^2(\Omega) = T\nabla\mathring{\mathrm{H}}^1(\Omega) \oplus_{T^{-1}} \ker\operatorname{div}$ from Theorem 2.6 we know that $\operatorname{ran}(\operatorname{div} T\mathring{\nabla}) = \operatorname{ran}(\operatorname{div}) = \mathrm{L}^2(\Omega)$. For $x = \left(\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right)$ for arbitrary $x_1 \in \{f \in \mathring{\mathrm{H}}^1(\Omega) \,|\, T\nabla x_1 \in \mathrm{H}(\operatorname{div}, \Omega)\}$ we have $x \in \operatorname{dom} A$ and by (9)

$$\langle \tfrac{1}{\rho} \operatorname{div} T\nabla x_1, y_2\rangle_{\mathrm{L}^2(\Omega)} = \langle T\nabla x_1, \nabla z_1\rangle_{\mathrm{L}^2(\Omega)} + 0 = -\langle \operatorname{div} T\nabla x_1, z_1\rangle_{\mathrm{L}^2(\Omega)}.$$

Thus, surjectivity of $\operatorname{div} T\mathring{\nabla}$ implies $z_1 = -\frac{1}{\rho} y_2$, in particular $\frac{1}{\rho} y_2 \in \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$ (because $z_1 \in \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$ by assumption).
Altogether we have shown $y \in \operatorname{dom} A$ and $z = -Ay$, i.e., $A^* \subseteq -A$
Note that A^* is not multi-valued as $A^* \subseteq -A$. This implies $(\operatorname{dom} A)^\perp = \{0\}$, cf. [19, Lem. 2.2.8] or equivalently $\overline{\operatorname{dom} A} = X$. ❑

With this, all initially outlined steps for verification of a boundary triple are in place.

Theorem 3.6 (Boundary triple)**.** $\big(\mathrm{H}^{-\frac{1}{2}}(\tilde\Gamma), \mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde\Gamma), B_1, B_2\big)$ *is a boundary triple for* A.

Proof. By definition of a boundary triple, we have to check, that an abstract Green identity holds (verified in Proposition 3.4) and that B is onto (verified in Corollary 3.3). ❑

[12] Alternatively one can set $A_0 \coloneqq -A$ with $\operatorname{dom} A_0 \coloneqq \ker B_1 \cap \ker B_2$, cf. [19, Lem. 2.4.5]. However, this requires careful handling to avoid circular arguments.

The remainder of this article heavily relies on exploiting Theorem 3.6. The first step is verifying well-posedness of the initial problem (1).

Well-posedness of Equation (1). For this we simply have to specify the boundary relation Θ from Equation (6). We set

$$\Theta := k_2 \in \mathcal{L}_{\mathrm{b}}(\mathrm{L}^2(\tilde{\Gamma})) \text{ (as an operator).}$$

Then the collection of boundary conditions from (3) can be written as $B_1 x + k_2 B_2 x = 0$ on $\tilde{\Gamma}$. We use the notation Θ just to fit the usual parameter for boundary triples. In particular

$$B_1 x + k_2 B_2 x = 0 \qquad \Longleftrightarrow \qquad k_1 w + \nu \cdot T\nabla w + k_2 \partial_t w = 0.$$

If one replaces $k_1, k_2 \in \mathcal{L}(\mathrm{L}^2(\tilde{\Gamma}))$ with suitable multiplication operators arising from functions by the same name, the equation to the right contains the second to the fifth condition of (3) in one line by taking into account that k_1 and k_2 may vanish on certain parts of $\tilde{\Gamma}$. The condition $w = 0$ on Γ_0 is already encoded in the domain of the operator A and completes the set of equations from (3). Hence, the operator that encodes all boundary conditions is A_Θ defined by (6) (with $A_0^* = A$ from (7) and $\Theta = k_2$), i.e.,

$$A_\Theta = \begin{pmatrix} 0 & \frac{1}{\rho} \\ \operatorname{div} T\nabla & 0 \end{pmatrix}, \quad \operatorname{dom} A_\Theta = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \operatorname{dom} A \,\middle|\, B_1 x + k_2 B_2 x = 0 \right\}.$$

Theorem 3.7. *A_Θ is generator of a strongly continuous semigroup of contractions.*

Proof. By Theorem 3.6 $\big((\mathrm{H}^{-\frac{1}{2}}(\tilde{\Gamma}), \mathring{\mathrm{H}}^{\frac{1}{2}}(\tilde{\Gamma})), B_1, B_2\big)$ is a boundary triple for A. Hence, Proposition 2.8 and the Lumer–Philips theorem (Theorem 2.5) imply the claim. ❑

Theorem 3.7 assures well-posedness of the wave equation without the perturbation terms a and b. Note that we assumed the initial conditions be in the state space X already in Section 1. For the full problem Equation (1) we define the perturbation

$$S \colon X \to X, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -a x_1 - \frac{1}{\rho} b x_2 \end{pmatrix}$$

for $a, b \in \mathrm{L}^\infty(\Omega)$ we obtain a bounded linear operator. Then Equation (1) becomes

$$\dot{x} = (A_\Theta + S)x$$

and we have unique solvability appealing to standard perturbation theory (e.g., [9, Thm. 1.3]). One can also allow more general a, b as long as S remains suitably relatively A_Θ bounded, cf. [9, Thm. 2.7] for a suitable perturbation result.

4. Stability

Under mild additional assumptions we are able to show stability of solutions. For the purpose of this section, we will additionally assume that:

- $\Gamma_0 \neq \partial\Omega$ and that there exists $\Gamma \subseteq \partial\Omega \setminus \overline{\Gamma_0}$ open and non-empty such that $k_2 > 0$ on Γ (i.e., there is an open set where the wave equation is damped).
- T and ρ are Lipschitz continuous multiplication operators.

Making use of the results of the previous section, we have to show stability properties of the semigroup generated by A_Θ. The notion of stabiliy we have in mind is the following one:

Definition 4.1. A strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator G is called *semi-uniformly stable*, if and only if there exists a continuous non-increasing function $f \colon [0,\infty) \to [0,\infty)$ satisfying $\lim_{t\to\infty} f(t) = 0$ such that for every $x \in \operatorname{dom} G$:

$$\lim_{t\to\infty} \|T(t)x\|_H \leq f(t) \|x\|_{\operatorname{dom} G}.$$

Remark 4.2. The notion of semi-uniform stability is nested in between strong stability where

$$\lim_{t\to\infty} \|T(t)x\|_H = 0$$

for all $x \in H$ is demanded and uniform stability, where

$$\lim_{t\to\infty} \|T(t)\| = 0$$

is demanded, which for strongly continuous semigroups is equivalent to uniform exponential stability, i.e., there exists $\epsilon > 0$ such that:

$$\lim_{t\to\infty} \mathrm{e}^{\epsilon t}\|T(t)\| = 0.$$

We rely on the following criterium from [2, Thm. 1.1][13].

Proposition 4.3. *Let $(T(t))_{t\geq 0}$ be a bounded strongly continuous semigroup with generator G satisfying $\sigma(G) \cap \mathrm{i}\mathbb{R} = \emptyset$. Then $(T(t))_{t\geq 0}$ is semi-uniformly stable.*

Proposition 4.3 outlines a roadmap we can follow with the goal of showing that A_Θ generates a semi-uniformly stable semigroup. We will verify:

(i) $(T(t))_{t\geq 0}$ is bounded.
(ii) $\mathrm{i}\lambda - A_\Theta$ is boundedly invertible for $\lambda \in \mathbb{R} \setminus \{0\}$.
(iii) A_Θ is boundedly invertible.

The point (i) is clear, as Proposition 2.8 states that A_Θ generates a contraction semigroup, in particular the semigroup is bounded. For (ii) and (iii) we follow the approach of [13]. We start by making the following observation:

Proposition 4.4. *The embedding of* $\operatorname{dom} A$ *into* X *is compact, i.e.,* $\operatorname{dom} A \overset{\mathsf{cpt}}{\hookrightarrow} X$.

Proof. Let $(\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right))_{n\in\mathbb{N}}$ be a bounded sequence in $\operatorname{dom} A$ w.r.t. the graph norm of A, i.e., there exists a $C > 0$ independent of $n \in \mathbb{N}$ such that

$$\|\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)\|^2_{\operatorname{dom} A} \coloneqq \|\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)\|^2_X + \|A\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)\|^2_X \leq C.$$

By the definition of the norm in X we obtain

$$\begin{aligned}\left\|\tfrac{1}{\rho}y_n\right\|^2_{\mathrm{L}^2(\Omega)} + \left\|T^{\frac{1}{2}}\nabla x_n\right\|^2_{\mathrm{L}^2(\Omega)} + \left\|k_1^{\frac{1}{2}}\gamma_0 x_n\right\|^2_{\mathrm{L}^2(\tilde{\Gamma})} \\ + \left\|\tfrac{1}{\rho}\operatorname{div} T\nabla x_n\right\|^2_{\mathrm{L}^2(\Omega)} + \left\|T^{\frac{1}{2}}\nabla \tfrac{1}{\rho}y_n\right\|^2_{\mathrm{L}^2(\Omega)} + \left\|k_1^{\frac{1}{2}}\gamma_0 \tfrac{1}{\rho}y_n\right\|^2_{\mathrm{L}^2(\tilde{\Gamma})} \leq C.\end{aligned}$$

This immediately implies that $(x_n)_{n\in\mathbb{N}}$ and $(\frac{1}{\rho}y_n)_{n\in\mathbb{N}}$ are bounded in $\mathrm{H}^1(\Omega)$.

We have to show that $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$ and $(y_n)_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathrm{L}^2(\Omega)$.

- Since $(\frac{1}{\rho}y_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{H}^1(\Omega)$, there exists a subsequence that converges in $\mathrm{L}^2(\Omega)$. Hence, $(\rho\frac{1}{\rho}y_{n(k)})_{k\in\mathbb{N}}$ converges in $\mathrm{L}^2(\Omega)$ as well.
- Again since $(x_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{H}^1(\Omega)$, there exists a subsequence that converges to x in $\mathrm{L}^2(\Omega)$. W.l.o.g. we pass to that subsequence. To avoid the introduction of several (irrelevant) multiplicative constants, we use the symbol $\lesssim$ which stands for inequality up to a multiplicative constant independent of n. Note that appealing to the boundary condition we have

$$\|\gamma_\nu T x_n\|_{\mathrm{L}^2(\tilde{\Gamma})} = \|k_1\gamma_0 x_n + k_2\gamma_0\tfrac{1}{\rho}y_n\|_{\mathrm{L}^2(\tilde{\Gamma})} \lesssim C.$$

[13]The decay rate can be made explicit, cf. [6, Thm. 3.4] and its proof for details.

Integration by parts and the Cauchy–Schwarz inequality give

$$\begin{aligned}
&\|\nabla(x_n - x_m)\|^2_{\mathrm{L}^2(\Omega)} \\
&\quad\lesssim \left\|T^{\frac{1}{2}}\nabla(x_n - x_m)\right\|^2 = \left\langle T\nabla(x_n - x_m), \nabla(x_n - x_m)\right\rangle \\
&\overset{\text{Int. by parts}}{=} \left\langle -\operatorname{div} T\nabla(x_n - x_m), x_n - x_m\right\rangle + \langle \gamma_\nu T(x_n - x_m), \gamma_0(x_n - x_m)\rangle \\
&\overset{\text{C.–S. ineq.}}{\lesssim} C\|x_n - x_m\|_{\mathrm{L}^2(\Omega)} + C\|\gamma_0(x_n - x_m)\|_{\mathrm{L}^2(\tilde{\Gamma})},
\end{aligned}$$

which shows that $(\nabla x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^2(\Omega)$ and hence convergent. By the closedness of the operator ∇, we conclude that x is also the limit of $(x_n)_{n\in\mathbb{N}}$ in $\mathrm{H}^1(\Omega)$. Finally, closedness of $\mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$ in $\mathrm{H}^1(\Omega)$ implies $x \in \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$. ❑

This result has the consequence that we only have to investigate eigenvalues:

Theorem 4.5. $\sigma(A_\Theta)$ *consists purely of eigenvalues.*

Proof. Let $\lambda - A_\Theta$ be injective. By Proposition 4.4 all resolvent operators of A_Θ are compact. Hence, let μ be such that $(\mu - A_\Theta)^{-1}$ is bijective and bounded (and compact). Then:

$$\lambda - A_\Theta = (\lambda - \mu) + (\mu - A_\Theta)$$
$$(\lambda - A_\Theta)(\mu - A_\Theta)^{-1} = (\lambda - \mu)(\mu - A_\Theta)^{-1} + 1$$

The right hand side is of the form $1 + K$, where K is a compact operator. It is a consequence of the theorem of Riesz-Schauder, cf. [24, Thm. 6.2.1], that an operator $1 + K$ is injective if and only if it is surjective. Now note that the left hand side is injective, since $\lambda - A_\Theta$ is injective and the resolvent is bijective. Thus $\lambda - A_\Theta$ is surjective. The open mapping theorem assures that $\lambda - A_\Theta$ is boundedly invertible. ❑

Proposition 4.3 requires us to study the spectrum of A_Θ on the imaginary axis. Theorem 4.5 makes this considerably easier, as we only have to check for eigenvalues. The following lemma gives us important information on the eigenvalues:

Lemma 4.6. *If* $\lambda \in \mathrm{i}\mathbb{R}$ *is an eigenvalue of* A_Θ*, the corresponding eigenvectors* x *satisfy*

$$\gamma_\nu T\nabla x_1 + k_1\gamma_0 x_1 = 0 \quad \text{and} \quad k_2\gamma_0\tfrac{1}{\rho}x_2 = 0.$$

Proof. We can calculate for $\lambda \in \sigma(A_\Theta)$ and $x \in \operatorname{dom} A_\Theta$ a corresponding eigenvector

$$\begin{aligned}
0 = \operatorname{Re}\big\langle \underbrace{(A - \lambda)x}_{=0}, x\big\rangle &= \operatorname{Re}\langle Ax, x\rangle - \operatorname{Re}\lambda\langle x, x\rangle \\
&= \tfrac{1}{2}\left(\langle Ax, x\rangle + \langle x, Ax\rangle\right) - \operatorname{Re}\lambda\|x\|^2 \\
&= \tfrac{1}{2}\left(\langle B_1 x, B_2 x\rangle_\mp + \langle B_2 x, B_1 x\rangle_\pm\right) - \operatorname{Re}\lambda\|x\|^2 \\
&= \operatorname{Re}\langle B_1 x, B_2 x\rangle_\mp - \operatorname{Re}\lambda\|x\|^2 \\
&= \operatorname{Re}\langle \gamma_\nu T\nabla x_1 + k_1\gamma_0 x_1, \gamma_0\tfrac{1}{\rho}x_2\rangle_\mp - \operatorname{Re}\lambda\|x\|^2.
\end{aligned}$$

If $\lambda \in \mathrm{i}\mathbb{R}$, then we obtain

$$0 = \operatorname{Re}\langle \gamma_\nu T\nabla x_1 + k_1\gamma_0 x_1, \gamma_0\tfrac{1}{\rho}x_2\rangle_\mp.$$

Appealing to the boundary condition and the fact that the duality can be written as an inner product if the arguments are in L^2 we infer

$$0 = \operatorname{Re}\langle k_2\gamma_0\tfrac{1}{\rho}x_2, \gamma_0\tfrac{1}{\rho}x_2\rangle_\mp = \operatorname{Re}\langle k_2\gamma_0\tfrac{1}{\rho}x_2, \gamma_0\tfrac{1}{\rho}x_2\rangle = \|k_2^{\frac{1}{2}}\gamma_0\tfrac{1}{\rho}x_2\|^2,$$

which implies $k_2\gamma_0\frac{1}{\rho}x_2 = 0$. Employing the boundary condition again we deduce $\gamma_\nu T\nabla x_1 + k_1\gamma_0 x_1 = 0$. ❑

Note that the previous result says that $\gamma_0 \frac{1}{\rho} x_2 = 0$ on $\operatorname{supp} k_2$, which by assumption (A6) implies that there is a non-empty open set Γ contained in $\operatorname{supp} k_2$ such that $\gamma_0 \frac{1}{\rho} x_2 = 0$ on Γ.
To study purely imaginary eigenvalues, we need to investigate the equation $(\mathrm{i}\lambda - A_\Theta)u = 0$, which can be written as the system:

$$\begin{aligned} \mathrm{i}\lambda u_1 - \tfrac{1}{\rho} u_2 &= 0 \quad \text{in } \mathrm{H}^1(\Omega) \\ \mathrm{i}\lambda u_2 - \operatorname{div} T \nabla u_1 &= 0 \quad \text{in } \mathrm{L}^2(\Omega). \end{aligned} \tag{10}$$

We consider the cases $\lambda \in \mathbb{R} \setminus \{0\}$ and $\lambda = 0$.

Lemma 4.7. $\mathrm{i}\lambda - A_\Theta$ *is boundedly invertible for* $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof. Let $\lambda \in \mathbb{R} \setminus \{0\}$ and let us assume that $\left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right)$ is an eigenvector to the eigenvalue λ. Then from Equation (10) we obtain that $u_1, u_2 \in \mathrm{H}^1(\Omega)$ and plugging the second equation into the first we obtain that $u_1 \in \mathrm{H}^1(\Omega)$ has to satisfy

$$\lambda^2 \rho u_1 + \operatorname{div} T \nabla u_1 = 0$$

Combining the first equation of (10) and Lemma 4.6 gives

$$\gamma_0 u_1 = 0 \quad \text{and} \quad \gamma_\nu T \nabla u_1 = 0 \quad \text{on} \quad \operatorname{supp} k_2.$$

Then the unique continuation principle[14], cf. Appendix A, implies that $u_1 = 0$ is the only solution, which in turn implies $u_2 = 0$ and therefore $u = 0$, which is a contradiction. ❑

Lemma 4.8. A_Θ *is boundedly invertible.*

Proof. Since $\operatorname{dom} A_\Theta \overset{\mathrm{cpt}}{\hookrightarrow} X$ by Proposition 4.4 it suffices to show that 0 is not an eigenvalue. Suppose 0 is an eigenvalue and $x \in \operatorname{dom} A_\Theta$ a corresponding eigenvector. Then from Equation (10) we immediately see $x_2 = 0$. Moreover, by Lemma 4.6

$$\gamma_\nu T \nabla x_1 + k_1 \gamma_0 x_1 = 0.$$

We test the equation $0 = \operatorname{div} T \nabla x_1$ with x_1 and integrate by parts:

$$0 = \langle \operatorname{div} T \nabla x_1, x_1 \rangle = -\langle T \nabla x_1, \nabla x_1 \rangle + \langle \gamma_\nu T \nabla x_1, \gamma_0 x_1 \rangle.$$

Utilizing the boundary condition $\gamma_\nu T \nabla x_1 + k \gamma_0 x_1 = 0$ we obtain

$$\left\langle \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \right\rangle_X = \left\langle \tfrac{1}{\rho} x_2, x_2 \right\rangle_{\mathrm{L}^2(\Omega)} + \langle T \nabla x_1, \nabla x_1 \rangle_{\mathrm{L}^2(\Omega)} + \langle k_1 \gamma_0 x_1, \gamma_0 x_1 \rangle_{\mathrm{L}^2(\tilde{\Gamma})} = 0.$$

Thus $x_1 = 0$ as well, but $x = 0$ is not an eigenvector. Hence, 0 cannot be an eigenvalue. ❑

Finally, we arrive at the stability result.

Theorem 4.9. A_Θ *generates a semi-uniformly stable semigroup.*

Proof. By Theorem 3.7 A_Θ generates a contraction semigroup. Theorem 4.5 shows that $\sigma(A_\Theta)$ consists only of eigenvalues, Lemma 4.7 and Lemma 4.8 show that the imaginary axis is contained in the resolvent set of A_Θ. Proposition 4.3 then implies the claim. ❑

Remark 4.10. We can also obtain semi-uniform stability of Equation (1) under suitable assumptions on the perturbation

$$S \colon X \supseteq \operatorname{dom}(S) \to X, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -a x_1 - b \frac{1}{\rho} x_2 \end{pmatrix}.$$

We see from Proposition 4.3 that it suffices for at least suitably A_Θ-bounded S (to assure that $A_\Theta + S$ still generates a semigroup of contractions) that S be negative.

[14]It is here that our stronger assumptions from the beginning of the section come into play. They are not required anywhere else!

5. Conclusion

As far as well-posedness of Equation (1) is concerned, we succeeded by means of a semigroup approach utilizing the theory of boundary triples; specifically, by constructing a boundary triple for the wave equation in the Lagrangian representation. This framework allows us to formulate boundary conditions that involve the displacement, its velocity and its normal stress. In particular we showed that the proposed boundary conditions induce a maximally dissipative operator. Well-posedness of Equation (1) is then a simple consequence of the Lumer–Philips theorem (and simple perturbation theory).

For our results on stability we first point out, that the situation covered in this article does not provide access to the tools usually employed in the verification of stability of solutions, which in most instances means exponential stability. This is because of the simple fact, that in our case, the damping happens on the boundary. Because of that lack of stronger techniques, we cannot show exponential stability and an application of the Gearhart–Prüss theorem seems out of reach, as we would need to prove sufficient resolvent estimates. The notion of "semi-uniform stability" presents a way out, with a convenient criterium (Proposition 4.3) requiring simple spectral theory. From the fact, that the domain of our differential operator is compactly embedded into the state space (Proposition 4.4) we immediately can conclude that its spectrum is a pure point-spectrum, making the spectral condition of Proposition 4.3 relatively easy to check. We point out, that the bottleneck for the stability part is the application of a unique continuation theorem to prove that the invariant system Equation (10) with zero boundary only admits the trivial solution. Only in this instance do we require Lipschitz continuity of T and ρ.

Appendix A. Unique continuation

In this section we want to briefly explain how we apply the unique continuation principle. In particular we use the version from [10]. To compare the notation we have $A = T$, $\mathbf{b} = 0$ and $V = \lambda^2 \rho$ (left hand side of the equality sign is the notation from [10]).

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded and connected Lipschitz domain, $\Gamma \subseteq \partial\Omega$ open (in $\partial\Omega$) and $w \in \mathrm{H}^1(\Omega)$ be a (weak) solution of

$$\begin{aligned} \operatorname{div} T\nabla w + \lambda^2 \rho w &= 0 \quad \text{in } \Omega, \\ \nu \cdot T\nabla w &= 0 \quad \text{on } \Gamma, \\ w &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Then we define for every $x \in \Omega$ the set Ω_x as in Figure 1, i.e., we choose a smooth path from x to a point outside of Ω that crosses $\partial\Omega$ in (the interior of) Γ and we define Ω_x as a smooth neighborhood of this path that is sufficiently small. Furthermore, we define

$$w_x(\zeta) \coloneqq \begin{cases} w(\zeta), & \zeta \in \Omega \cap \Omega_x, \\ 0, & \zeta \in \Omega_x \setminus \Omega. \end{cases}$$

Note that we can extend T and ρ Lipschitz continuously and boundedly to all of $\mathbb{R}^d$.[15] By splitting the area of integration, using integration by parts and the boundary conditions on Γ we can see that $w_x \in \mathrm{H}^1(\Omega_x)$ such that $T\nabla w_x \in \mathrm{H}(\operatorname{div}, \Omega_x)$. In particular, this leads to w_x solving

$$\operatorname{div} T\nabla w + \lambda^2 \rho w = 0 \quad \text{in } \Omega_x.$$

[15] We can Lipschitz continuously extend T to a neigborhood of Ω and we then work with $\alpha T + (1-\alpha)\mathrm{I}$, where α is a cutoff function. This construction gives an extension that is globally bounded. Analogously, we can extend ρ.

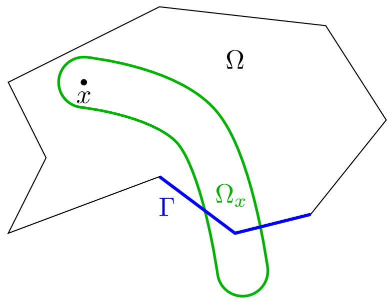

Figure 1. Deriving boundary unique continuation from strong unique continuation.

We aim to apply the unique continuation principle [10, Thm. 1.1] to conclude that w_x is zero. Note that the assumptions of [10, Thm. 1.1] are easy to check for our setting as $\mathbf{b} = 0$ and $V = \lambda^2\rho$ is bounded. Hence, $w_x = 0$ and consequently also $w = 0$.

Appendix B. Regularity

Note that for general (non-Lipschitz continuous) T the C^∞ functions are not necessarily dense in the domain of $\operatorname{div} T\nabla$, as they may not even lie in the domain. Hence, arguments that rely on smooth functions in the domain of $\operatorname{div} T\nabla$ and extend these properties by density require an alternative dense set of regular functions. In a previous version we relied on such arguments, but we could find more direct methods. Nevertheless, we aim to identify such a set of "regular" functions that is dense for potential future applications. In particular we are concerned with boundary regularity—specifically, the existence of a meaningful $\mathrm{L}^2(\partial\Omega)$ normal trace for $T\nabla w$. We therefore define the following spaces

$$\begin{aligned} \mathring{\mathrm{H}}(\operatorname{div} 0,\Omega) &\coloneqq \{u \in \mathrm{H}(\operatorname{div},\Omega) \,|\, \operatorname{div} u = 0, \gamma_\nu u = 0\} \\ \hat{\mathrm{H}}(\operatorname{div},\Omega) &\coloneqq \{u \in \mathrm{H}(\operatorname{div},\Omega) \,|\, \gamma_\nu u \in \mathrm{L}^2(\partial\Omega)\}. \end{aligned}$$

We will disregard the additional boundary condition on Γ_0 imposed by $\mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$ (i.e., we regard $\Gamma_0 = \emptyset$) as it actually simplifies the argument rather than complicating it, because in this space $\|\nabla\cdot\|_{\mathrm{L}^2(\Omega)}$ is equivalent to the full $\mathrm{H}^1(\Omega)$ norm. The only part that needs more attention in that case is the Helmholtz decomposition, but this is covered by [3, Thm. 5.3].

Theorem B.1. *The set* $D_1 \coloneqq \{f \in \mathrm{H}^1(\Omega) \,|\, T\nabla f \in \hat{\mathrm{H}}(\operatorname{div},\Omega)\}$ *is a core of* $L = \operatorname{div} T\nabla$ *(with* $\operatorname{dom} L = \{f \in \mathrm{H}^1(\Omega) \,|\, T\nabla f \in \mathrm{H}(\operatorname{div},\Omega)\}$ *as operator on* $\mathrm{L}^2(\Omega)$*).*

Proof. Let $f \in \operatorname{dom} L$. Note that $\hat{\mathrm{H}}(\operatorname{div},\Omega)$ is dense in $\mathrm{H}(\operatorname{div},\Omega)$ w.r.t. $\|\cdot\|_{\mathrm{H}(\operatorname{div},\Omega)}$. Hence, for given $\epsilon > 0$ there exists a $g \in \hat{\mathrm{H}}(\operatorname{div},\Omega)$ such that $\|T\nabla f - g\|_{\mathrm{H}(\operatorname{div},\Omega)} \leq \epsilon$. Moreover, similar to Theorem 2.6, by replacing $\mathring{\nabla}$, the gradient with Dirichlet boundary, with ∇, as in the proof there we can decompose $\mathrm{L}^2(\Omega)$ into

$$\mathrm{L}^2(\Omega) = T\nabla \mathrm{H}^1(\Omega) \oplus_{T^{-1}} \mathring{\mathrm{H}}(\operatorname{div} 0,\Omega),$$

where $a \perp_{T^{-1}} b$ means $\langle T^{-1}a, b\rangle_{\mathrm{L}^2(\Omega)} = 0$. Hence, there exist $h \in \mathrm{H}^1(\Omega)$ and $k \in \mathring{\mathrm{H}}(\operatorname{div} 0,\Omega)$ such that $g = T\nabla h + k$. Since $g, k \in \hat{\mathrm{H}}(\operatorname{div},\Omega)$ we conclude that also $T\nabla h \in \hat{\mathrm{H}}(\operatorname{div},\Omega)$. Moreover,

$$\gamma_\nu g = \gamma_\nu T\nabla h + \underbrace{\gamma_\nu k}_{=0} = \gamma_\nu T\nabla h$$

and

$$\operatorname{div} g = \operatorname{div} T\nabla h + \underbrace{\operatorname{div} k}_{=0} = \operatorname{div} T\nabla h.$$

This gives $\|\operatorname{div} T\nabla(f-h)\|_{\mathrm{L}^2(\Omega)} = \|\operatorname{div} T\nabla f - g\|_{\mathrm{L}^2(\Omega)} \le \epsilon$. Clearly, we also have $\|T\nabla f - g\|_{\mathrm{L}^2(\Omega)} \le \epsilon$. Moreover, by orthogonality we obtain

$$\begin{aligned}\epsilon^2 &\ge \|T\nabla f - T\nabla h - k\|^2_{\mathrm{L}^2(\Omega)} \ge \frac{1}{\|T^{\frac{1}{2}}\|^2}\|T\nabla(f-h) - k\|^2_{T^{-1}}\\ &= \frac{1}{\|T^{\frac{1}{2}}\|^2}\big(\|T\nabla(f-h)\|^2_{T^{-1}} + \|k\|^2_{T^{-1}}\big),\end{aligned}$$

which implies $\|T\nabla(f-h)\|_{T^{-1}} \le \|T^{-\frac{1}{2}}\|\epsilon$ and consequently $\|T\nabla(f-h)\|_{\mathrm{L}^2(\Omega)} \le \|T^{\frac{1}{2}}\|\|T^{-\frac{1}{2}}\|\epsilon$. Hence, we have

$$\|T\nabla(f-h)\|_{\mathrm{H}(\operatorname{div},\Omega)} \le (1 + \|T^{\frac{1}{2}}\|\|T^{-\frac{1}{2}}\|)\epsilon.$$

Moreover, we have the decomposition

$$\mathrm{H}^1(\Omega) = \mathbb{C}^\perp \oplus \mathbb{C}$$

and that $\nabla\colon \mathbb{C}^\perp \to \operatorname{ran}\nabla$ is a boundedly invertible mapping, where $\operatorname{ran}\nabla$ is endowed with $\|\cdot\|_{\mathrm{L}^2(\Omega)}$. Note that f and h can be decomposed according to this decomposition into $f = f_1 + f_2$ and $h = h_1 + h_2$, where $f_1, h_1 \in \mathbb{C}^\perp$ and $f_2, h_2 \in \mathbb{C}$. Since T is also boundedly invertible we obtain

$$\|f_1 - h_1\|_{\mathrm{L}^2(\Omega)} = \|\nabla^{-1}T^{-1}T\nabla(f-h)\|_{\mathrm{L}^2(\Omega)} \le C\|T^{\frac{1}{2}}\|\|T^{-\frac{1}{2}}\|\epsilon.$$

Hence, we define $\phi = h_1 + f_2 \in \mathrm{H}^1(\Omega)$, which implies $\nabla\phi = \nabla h$ and in particular $\phi \in D_1$. This gives

$$\begin{aligned}\|f-\phi\|_{\operatorname{dom} L} &= \sqrt{\|f-\phi\|^2_{\mathrm{L}^2(\Omega)} + \|\operatorname{div} T\nabla(f-\phi)\|^2_{\mathrm{L}^2(\Omega)}}\\ &= \sqrt{\|f_1-h_1\|^2_{\mathrm{L}^2(\Omega)} + \|\operatorname{div} T\nabla(f-h)\|^2_{\mathrm{L}^2(\Omega)}} \le \tilde{C}\epsilon\end{aligned}$$

for a constant $C > 0$. Note that by construction $\phi \in D_1$ and $\|f-\phi\|_{\operatorname{dom} L} \le \hat{C}\epsilon$, which proves that D_1 is a core of L. ❑

Consequently for $A = \begin{pmatrix} 0 & \frac{1}{\rho} \\ \operatorname{div} T\nabla & 0\end{pmatrix}$ with $\Gamma_0 = \emptyset$ from Section 3 we obtain:

Corollary B.2. *The set*

$$D := \left\{\begin{pmatrix} x_1 \\ x_2\end{pmatrix} \in \operatorname{dom}(A) \,\middle|\, \gamma_\nu T\nabla x_1 \in \mathrm{L}^2(\partial\Omega)\right\} = D_1 \times \rho\mathrm{H}^1(\Omega)$$

is a core of A.

Remark B.3. Note the general statement with non-empty Γ_0 can be obtained relatively similar. The only difference is that in the proof of Theorem B.1 we need the corresponding Helmholtz decomposition

$$\mathrm{L}^2(\Omega) = T\nabla\mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega) \oplus_{T^{-1}} \mathring{\mathrm{H}}_{\partial\Omega\setminus\Gamma_0}(\operatorname{div} 0, \Omega),$$

see, e.g., [3, Thm. 5.3].

Appendix C. Friedrichs/Poincaré inequality

We use a slightly modified version of Friedrichs/Poincaré's inequality.

Theorem C.1. *Let Ω be a bounded and connected Lipschitz domain, $\Gamma \subseteq \partial\Omega$ with positive measure and $k_1 \in \mathcal{L}_\mathrm{b}(\mathrm{L}^2(\Gamma))$ such that $1 \notin \ker k_1$. Then there exists a $C > 0$ such that for all $f \in \mathrm{H}^1(\Omega)$*

$$\|f\|_{\mathrm{L}^2(\Omega)} \le C\big(\|\nabla f\|_{\mathrm{L}^2(\Omega)} + \big\|k_1^{\frac{1}{2}}\gamma_0 f\big\|_{\mathrm{L}^2(\Gamma)}\big).$$

Proof. Assume that there is no such $C > 0$. Then we find a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{H}^1(\Omega)$ such that

$$\|f_n\|_{\mathrm{L}^2(\Omega)} > n\big(\|\nabla f_n\|_{\mathrm{L}^2(\Omega)} + \big\|k_1^{\frac{1}{2}}\gamma_0 f_n\big\|_{\mathrm{L}^2(\Gamma)}\big).$$

We define $g_n := \frac{f_n}{\|f_n\|_{\mathrm{L}^2(\Omega)}}$, which implies

$$\|g_n\|_{\mathrm{L}^2(\Omega)} = 1, \quad \|\nabla g_n\|_{\mathrm{L}^2(\Omega)} \to 0 \quad \text{and} \quad \|k_1^{\frac{1}{2}}\gamma_0 g_n\|_{\mathrm{L}^2(\Gamma)} \to 0.$$

By the Rellich–Kondrachov theorem and passing over to a subsequence the sequence $(g_n)_{n\in\mathbb{N}}$ converges to a $g \in \mathrm{L}^2(\Omega)$ (w.r.t. $\|\cdot\|_{\mathrm{L}^2(\Omega)}$). For $\phi \in \mathrm{C}_\mathrm{c}^\infty(\Omega)$ we have

$$\langle \operatorname{div}\phi, g\rangle_{\mathrm{L}^2(\Omega)} = \lim_{n\to\infty} \langle \operatorname{div}\phi, g_n\rangle_{\mathrm{L}^2(\Omega)} = \lim_{n\to\infty} -\langle \phi, \nabla g_n\rangle_{\mathrm{L}^2(\Omega)} = 0,$$

which implies that $g \in \mathrm{H}^1(\Omega)$ and $\nabla g = 0$. Consequently, g is constant, i.e., there exists a $c \in \mathbb{C}$ such that $g = c$. Moreover, the sequence $(g_n)_{n\in\mathbb{N}}$ converges also w.r.t. $\|\cdot\|_{\mathrm{H}^1(\Omega)}$ to g. This gives

$$k_1 c = k_1\gamma_0 g = \lim_{n\to\infty} k_1^{\frac{1}{2}} k_1^{\frac{1}{2}} \gamma_0 g_n = 0$$

and from the assumption on k_1 we conclude $c = 0$, which contradicts $\|g\|_{\mathrm{L}^2(\Omega)} = 1$. ❑

References

[1] K. Ammari, S. Nicaise, and C. Pignotti. Feedback boundary stabilization of wave equations with interior delay. *Systems Control Lett.*, 59(10):623–628, 2010. doi:10.1016/j.sysconle.2010.07.007.

[2] C. J. K. Batty and T. Duyckaerts. Non-uniform stability for bounded semi-groups on Banach spaces. *J. Evol. Equ.*, 8(4):765–780, 2008. doi:10.1007/s00028-008-0424-1.

[3] S. Bauer, D. Pauly, and M. Schomburg. The Maxwell compactness property in bounded weak Lipschitz domains with mixed boundary conditions. *SIAM J. Math. Anal.*, 48(4):2912–2943, 2016. doi:10.1137/16M1065951.

[4] J. Behrndt, S. Hassi, and H. de Snoo. *Boundary value problems, Weyl functions, and differential operators*, volume 108 of *Monographs in Mathematics*. Birkhäuser/Springer, Cham, [2020] ©2020. doi:10.1007/978-3-030-36714-5.

[5] A. Bendimerad-Hohl, G. Haine, L. Lefèvre, and D. Matignon. Stokes-lagrange and stokes-dirac representations of n-dimensional port-hamiltonian systems for modelling and control, 2024. URL: https://arxiv.org/abs/2412.04499, arXiv:2412.04499.

[6] R. Chill, D. Seifert, and Y. Tomilov. Semi-uniform stability of operator semigroups and energy decay of damped waves. *Phil. Trans. R. Soc. A*, 378, 2020. URL: http://dx.doi.org/10.1098/rsta.2019.0614, doi:10.1098/rsta.2019.0614.

[7] R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology. Volume 3: Spectral theory and applications. With the collaboration of Michel Artola and Michel Cessenat. Transl. from the French by John C. Amson.* Berlin: Springer, 2nd printing edition, 2000.

[8] M. Eller and I. M. Karabash. M-dissipative boundary conditions and boundary tuples for Maxwell operators. *J. Differential Equations*, 325:82–118, 2022. doi:10.1016/j.jde.2022.04.006.

[9] K.-J. Engel and R. Nagel. *A short course on operator semigroups*. Universitext. Springer, New York, 2006.

[10] N. Garofalo and F.-H. Lin. Unique continuation for elliptic operators: A geometric-variational approach. *Commun. Pure Appl. Math.*, 40(3):347–366, 1987. doi:10.1002/cpa.3160400305.

[11] H. Gernandt, F. Philipp, T. Preuster, and M. Schaller. Extension theory via boundary triplets for infinite-dimensional implicit port-hamiltonian systems, 2025. URL: https://arxiv.org/abs/2503.17874, arXiv:2503.17874.

[12] V. I. Gorbachuk and M. L. Gorbachuk. *Boundary value problems for operator differential equations*, volume 48 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991. Translated and revised from the 1984 Russian original. doi:10.1007/978-94-011-3714-0.

[13] B. Jacob and N. Skrepek. Stability of the multidimensional wave equation in port-hamiltonian modelling. In *2021 60th IEEE Conference on Decision and Control (CDC)*, pages 6188–6193, 2021. doi:10.1109/CDC45484.2021.9683501.

[14] B. Jacob and H. J. Zwart. *Linear port-Hamiltonian systems on infinite-dimensional spaces.*, volume 223 of *Oper. Theory: Adv. Appl.* Basel: Birkhäuser, 2012. doi:10.1007/978-3-0348-0399-1.

[15] M. Kurula and H. Zwart. Linear wave systems on n-D spatial domains. *Internat. J. Control*, 88(5):1063–1077, 2015. doi:10.1080/00207179.2014.993337.

[16] S. Nicaise, C. Pignotti, and J. Valein. Exponential stability of the wave equation with boundary time-varying delay. *Discrete Contin. Dyn. Syst. Ser. S*, 4(3):693–722, 2011. doi:10.3934/dcdss.2011.4.693.

[17] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Appl. Math. Sci.* Springer, Cham, 1983. doi:10.1007/978-1-4612-5561-1.

[18] C. Seifert, S. Trostorff, and M. Waurick. *Evolutionary equations—Picard's theorem for partial differential equations, and applications*, volume 287 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer, Cham, [2022] ©2022. doi:10.1007/978-3-030-89397-2.

[19] N. Skrepek. *Linear port-Hamiltonian Systems on Multidimensional Spatial Domains.* PhD thesis, University of Wuppertal, 2021. doi:10.25926/g7h8-bd50.

[20] N. Skrepek. Well-posedness of linear first order port-Hamiltonian systems on multidimensional spatial domains. *Evol. Equ. Control Theory*, 10(4):965–1006, 2021. doi:10.3934/eect.2020098.

[21] N. Skrepek and M. Waurick. Semi-uniform stabilization of anisotropic Maxwell's equations via boundary feedback on split boundary. *J. Differential Equations*, 394:345–374, 2024. doi:10.1016/j.jde.2024.03.021.

[22] S. Touhami, A. Chaira, and D. F. M. Torres. Functional characterizations of trace spaces in Lipschitz domains. *Banach J. Math. Anal.*, 13(2):407–426, 2019. doi:10.1215/17358787-2018-0044.

[23] S.-A. Wegner. Boundary triplets for skew symmetric operators and the generation of strongly continuous semigroups. *Anal. Math.*, 43(4):657–686, 2017. doi:10.1007/s10476-017-0509-6.

[24] D. Werner. *Funktionalanalysis.* Springer-Verlag, Berlin, extended edition, 2000.

[25] E. Zuazua. Exponential decay for the semilinear wave equation with locally distributed damping. *Comm. Partial Differential Equations*, 15(2):205–235, 1990. doi:10.1080/03605309908820684.

Institute of Applied Analysis, TU Bergakademie Freiberg, Akademiestrasse 6, 09599 Freiberg, Sachsen, Germany
Email address: bernhard.aigner@doktorand.tu-freiberg.de

Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, Overijssel, The Netherlands
Email address: n.skrepek@utwente.nl