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A compactness result for the div-curl system with inhomogeneous mixed boundary conditions for bounded Lipschitz domains and some applications

Dirk Pauly[1] · Nathanael Skrepek[2]

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Abstract
For a bounded Lipschitz domain with Lipschitz interface we show the following *compactness theorem*: Any L^2-bounded sequence of vector fields with L^2-bounded rotations and L^2-bounded divergences as well as L^2-bounded tangential traces on one part of the boundary and L^2-bounded normal traces on the other part of the boundary, contains a strongly L^2-convergent subsequence. This generalises recent results for homogeneous mixed boundary conditions in Bauer et al. (SIAM J Math Anal 48(4):2912-2943, 2016) Bauer et al. (in: Maxwell's Equations: Analysis and Numerics (Radon Series on Computational and Applied Mathematics 24), De Gruyter, pp. 77-104, 2019). As applications we present a related *Friedrichs/Poincaré type estimate*, a *div-curl lemma*, and show that the Maxwell operator with mixed tangential and impedance boundary conditions (Robin type boundary conditions) has *compact resolvents*.

Keywords Compact embeddings · Div-curl system · Mixed boundary conditions · Inhomogeneous boundary conditions

Mathematics Subject Classification 35A23 · 35Q61

Contents

✉ Dirk Pauly
dirk.pauly@tu-dresden.de

Nathanael Skrepek
nathanael.skrepek@math.tu-freiberg.de

1 Institut für Analysis, Technische Universität Dresden, Dresden, Germany

2 Institut für Angewandte Analysis, Technische Universität Bergakademie Freiberg, Freiberg, Germany

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1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be open with boundary Γ, composed of the boundary parts Γ_0 (tangential) and Γ_1 (normal). In [2, Theorem 4.7] the following version of Weck's selection theorem has been shown:

Theorem 1.1 (compact embedding for vector fields with homogeneous mixed boundary conditions) Let (Ω, Γ_0) be a bounded strong[1] Lipschitz pair and let ε be admissible[2]. Then

$$\mathsf{H}_{\Gamma_0}(\text{curl}, \Omega) \cap \varepsilon^{-1}\mathsf{H}_{\Gamma_1}(\text{div}, \Omega) \overset{\text{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega).$$

Here, $\overset{\text{cpt}}{\hookrightarrow}$ denotes a compact embedding, and—in classical terms and in the smooth case—we have for a vector field E (n denotes the exterior unit normal at Γ)

$$\begin{aligned} E &\in \mathsf{H}_{\Gamma_0}(\text{curl}, \Omega) &\Leftrightarrow&\quad E \in \mathsf{L}^2(\Omega), \quad \text{curl}\, E \in \mathsf{L}^2(\Omega), \quad n \times E|_{\Gamma_0} = 0, \\ E &\in \varepsilon^{-1}\mathsf{H}_{\Gamma_1}(\text{div}, \Omega) &\Leftrightarrow&\quad \varepsilon E \in \mathsf{L}^2(\Omega), \quad \text{div}\, \varepsilon E \in \mathsf{L}^2(\Omega), \quad n \cdot \varepsilon E|_{\Gamma_1} = 0. \end{aligned}$$

For exact definitions and notations see Sect. 2, and for a history of related compact embedding results see, e.g., [5, 7, 19, 21, 23, 24, 26] and [9]. The general importance of compact embeddings in a functional analytical setting (FA-ToolBox) for Hilbert complexes (such as de Rham, elasticity, biharmonic) is described, e.g., in [13–16] and [1, 17, 18].

In this paper, we shall generalise Theorem 1.1 to the case of inhomogeneous boundary conditions, i.e., we will show that the compact embedding in Theorem 1.1 still holds if the space

$$\mathsf{H}_{\Gamma_0}(\text{curl}, \Omega) \cap \varepsilon^{-1}\mathsf{H}_{\Gamma_1}(\text{div}, \Omega)$$

is replaced by

$$\widehat{\mathsf{H}}_{\Gamma_0}(\text{curl}, \Omega) \cap \varepsilon^{-1}\widehat{\mathsf{H}}_{\Gamma_1}(\text{div}, \Omega),$$

[1] Both Ω and the interface $\overline{\Gamma_0} \cap \overline{\Gamma_1}$ are locally defined by graphs of Lipschitz functions.

[2] See Sect. 2.

where in classical terms and in the smooth case

$$E \in \widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{curl}, \Omega) \quad \Leftrightarrow \quad E \in \mathsf{L}^2(\Omega), \quad \mathrm{curl}\, E \in \mathsf{L}^2(\Omega), n \times E|_{\Gamma_0} \in \mathsf{L}^2(\Gamma_0),$$
$$E \in \varepsilon^{-1}\widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div}, \Omega) \quad \Leftrightarrow \quad \varepsilon E \in \mathsf{L}^2(\Omega), \quad \mathrm{div}\, \varepsilon E \in \mathsf{L}^2(\Omega), n \cdot \varepsilon E|_{\Gamma_1} \in \mathsf{L}^2(\Gamma_1).$$

The main result (compact embedding) is formulated in Theorem 4.1. Note that while Theorem 1.1 even holds for bounded weak[3] Lipschitz pairs (Ω, Γ_0), cf. [2, Theorem 4.7], Theorem 4.1 is only shown for bounded strong Lipschitz pairs. This comes by using regular decompositions in Ω which fail in the weak Lipschitz case as—roughly speaking—the corresponding transformations respect $\mathsf{H}(\mathrm{curl}, \Omega)$ and $\mathsf{H}(\mathrm{div}, \Omega)$ regularity but not $\mathsf{H}^1(\Omega)$ regularity. Moreover, we emphasise that the additional L^2 regularity at the boundary is crucial since the natural $\mathsf{H}^{-1/2}$ regularity at the boundary does not allow for compact embeddings. E.g., it is well known that $\mathsf{H}(\mathrm{curl}, \Omega) \cap \mathsf{H}(\mathrm{div}, \Omega)$ is not compactly embedded into $\mathsf{L}^2(\Omega)$.

As applications we show that the compact embedding implies a related Friedrichs/Poincaré type estimate, cf. Theorem 5.1, showing well-posedness of related systems of partial differential equations. Moreover, in Theorem 5.3 we prove that Theorem 4.1 yields a div-curl lemma. Note that corresponding results for exterior domains are straight forward using weighted Sobolev spaces, see [11, 12]. Another application is presented in Sect. 5.3 where we show that our compact embedding result implies compact resolvents of the Maxwell operator with inhomogeneous mixed boundary conditions, even of impedance type. We finally note in Sect. 5.4 that the corresponding result holds (in the simpler situation) for the impedance wave equation (acoustics) as well.

2 Notations

Throughout this paper, let $\Omega \subset \mathbb{R}^3$ be an open and bounded strong Lipschitz domain, and let ε be an *admissible* tensor (matrix) field, i.e., a symmetric, L^∞-bounded, and uniformly positive definite tensor field $\varepsilon : \Omega \to \mathbb{R}^{3\times 3}$. Moreover, let the boundary Γ of Ω be decomposed into two relatively open and strong Lipschitz subsets Γ_0 and $\Gamma_1 := \Gamma \setminus \overline{\Gamma_0}$ forming the interface $\overline{\Gamma_0} \cap \overline{\Gamma_1}$ for the mixed boundary conditions. See [2–4] for exact definitions. We call (Ω, Γ_0) a bounded strong Lipschitz pair.

The usual Lebesgue and Sobolev Hilbert spaces (of scalar or vector valued fields) are denoted by $\mathsf{L}^2(\Omega)$, $\mathsf{H}^1(\Omega)$, $\mathsf{H}(\mathrm{curl}, \Omega)$, $\mathsf{H}(\mathrm{div}, \Omega)$, and by $\mathsf{H}_0(\mathrm{curl}, \Omega)$ and $\mathsf{H}_0(\mathrm{div}, \Omega)$ we indicate the spaces with vanishing curl and div, respectively. Homogeneous boundary conditions are introduced in the strong sense as closures of respective test fields from

$$\mathsf{C}^\infty_{\Gamma_0}(\Omega) := \Big\{\phi|_\Omega \, : \, \phi \in \mathsf{C}^\infty(\mathbb{R}^3), \ \mathrm{supp}\,\phi \text{ compact}, \ \mathrm{dist}(\mathrm{supp}\,\phi, \Gamma_0) > 0\Big\},$$

[3] Both Ω and the interface $\overline{\Gamma_0} \cap \overline{\Gamma_1}$ are Lipschitz submanifolds.

i.e.,

$$\mathsf{H}^1_{\Gamma_0}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_0}(\Omega)}^{\mathsf{H}^1(\Omega)}, \quad \mathsf{H}_{\Gamma_0}(\mathrm{curl}, \Omega) := \overline{\mathsf{C}^\infty_{\Gamma_0}(\Omega)}^{\mathsf{H}(\mathrm{curl},\Omega)},$$
$$\mathsf{H}_{\Gamma_0}(\mathrm{div}, \Omega) := \overline{\mathsf{C}^\infty_{\Gamma_0}(\Omega)}^{\mathsf{H}(\mathrm{div},\Omega)},$$

and we set $\mathsf{H}^1_\emptyset(\Omega) := \mathsf{H}^1(\Omega)$, $\mathsf{H}_\emptyset(\mathrm{curl}, \Omega) := \mathsf{H}(\mathrm{curl}, \Omega)$, and $\mathsf{H}_\emptyset(\mathrm{div}, \Omega) := \mathsf{H}(\mathrm{div}, \Omega)$. Spaces with vanishing curl and div are again denoted by $\mathsf{H}_{\Gamma_0,0}(\mathrm{curl}, \Omega)$ and $\mathsf{H}_{\Gamma_0,0}(\mathrm{div}, \Omega)$, respectively. Moreover, we introduce the cohomology space of Dirichlet/Neumann fields (generalised harmonic fields)

$$\mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega) := \mathsf{H}_{\Gamma_0,0}(\mathrm{curl}, \Omega) \cap \varepsilon^{-1}\mathsf{H}_{\Gamma_1,0}(\mathrm{div}, \Omega).$$

The $\mathsf{L}^2(\Omega)$-inner product and norm (of scalar or vector valued $\mathsf{L}^2(\Omega)$-spaces) will be denoted by $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2(\Omega)}$ and $\|\cdot\|_{\mathsf{L}^2(\Omega)}$, respectively, and the weighted Lebesgue space $\mathsf{L}^2_\varepsilon(\Omega)$ is defined as $\mathsf{L}^2(\Omega)$ (of vector fields) but being equipped with the weighted $\mathsf{L}^2(\Omega)$-inner product and norm $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega)} := \langle\varepsilon\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2(\Omega)}$ and $\|\cdot\|_{\mathsf{L}^2_\varepsilon(\Omega)}$, respectively. The norms in, e.g., $\mathsf{H}^1(\Omega)$ and $\mathsf{H}(\mathrm{curl}, \Omega)$ are denoted by $\|\cdot\|_{\mathsf{H}^1(\Omega)}$ and $\|\cdot\|_{\mathsf{H}(\mathrm{curl},\Omega)}$, respectively. Orthogonality and orthogonal sum in $\mathsf{L}^2(\Omega)$ and $\mathsf{L}^2_\varepsilon(\Omega)$ are indicated by $\perp_{\mathsf{L}^2(\Omega)}$, $\perp_{\mathsf{L}^2_\varepsilon(\Omega)}$, and $\oplus_{\mathsf{L}^2(\Omega)}$, $\oplus_{\mathsf{L}^2_\varepsilon(\Omega)}$, respectively.

Finally, we introduce inhomogeneous tangential and normal L^2-boundary conditions in

$$\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{curl}, \Omega) := \Big\{E \in \mathsf{H}(\mathrm{curl}, \Omega)\ :\ \tau_{\Gamma_0} E \in \mathsf{L}^2(\Gamma_0)\Big\},$$
$$\widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div}, \Omega) := \Big\{E \in \mathsf{H}(\mathrm{div}, \Omega)\ :\ \nu_{\Gamma_1} E \in \mathsf{L}^2(\Gamma_1)\Big\}$$

with norms given by, e.g., $\|E\|^2_{\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{curl},\Omega)} := \|E\|^2_{\mathsf{H}(\mathrm{curl},\Omega)} + \|\tau_{\Gamma_0}E\|^2_{\mathsf{L}^2(\Gamma_0)}$. The definitions of the latter Hilbert spaces and traces need some explanations:

Definition and Remark 2.1 (L^2-traces.)

(i) The tangential trace of a vector field $E \in \mathsf{H}(\mathrm{curl}, \Omega)$ is a well-defined tangential vector field $\tau_\Gamma E \in \mathsf{H}^{-1/2}(\Gamma)$ generalising the classical tangential trace $\tau_\Gamma \widetilde{E} = -n \times n \times \widetilde{E}|_\Gamma$ for smooth vector fields $\widetilde{E}$. By the notation $\tau_{\Gamma_0} E \in \mathsf{L}^2(\Gamma_0)$ we mean, that there exists a tangential vector field $E_{\Gamma_0} \in \mathsf{L}^2(\Gamma_0)$, such that for all vector fields $\Phi \in \mathsf{H}^1_{\Gamma_1}(\Omega)$ it holds

$$\langle \mathrm{curl}\,\Phi, E\rangle_{\mathsf{L}^2(\Omega)} - \langle \Phi, \mathrm{curl}\,E\rangle_{\mathsf{L}^2(\Omega)} = \langle \tau^\times_{\Gamma_0}\Phi, E_{\Gamma_0}\rangle_{\mathsf{L}^2(\Gamma_0)}.$$

Then we set $\tau_{\Gamma_0} E := E_{\Gamma_0} \in \mathsf{L}^2(\Gamma_0)$. Here and in the following, the twisted tangential trace of the smooth vector field Φ is given by the tangential vector field $\tau^\times_\Gamma \Phi = n \times \Phi|_\Gamma \in \mathsf{L}^2(\Gamma)$ with $\tau^\times_{\Gamma_1}\Phi = \tau^\times_\Gamma \Phi|_{\Gamma_1} = 0$ and $\tau^\times_{\Gamma_0}\Phi = \tau^\times_\Gamma \Phi|_{\Gamma_0} \in \mathsf{L}^2(\Gamma_0)$. Note that $\tau_{\Gamma_0} E$ is well defined as $\tau^\times_{\Gamma_0}\mathsf{H}^1_{\Gamma_1}(\Omega)$ is dense in $\mathsf{L}^2_t(\Gamma_0) = \big\{v \in \mathsf{L}^2(\Gamma_0) : n \cdot v = 0\big\}$.

(ii) Analogously, the normal trace of a vector field $E \in \mathsf{H}(\mathrm{div}, \Omega)$ is a well-defined function $\nu_\Gamma E \in \mathsf{H}^{-1/2}(\Gamma)$ generalising the classical normal trace $\nu_\Gamma \widetilde{E} = n \cdot \widetilde{E}|_\Gamma$ for smooth vector fields $\widetilde{E}$. Again, by the notation $\nu_{\Gamma_1} E \in \mathsf{L}^2(\Gamma_1)$ we mean, that for all functions $\phi \in \mathsf{H}^1_{\Gamma_0}(\Omega)$ it holds

$$\langle \nabla \phi, E \rangle_{\mathsf{L}^2(\Omega)} + \langle \phi, \mathrm{div}\, E \rangle_{\mathsf{L}^2(\Omega)} = \langle \sigma_{\Gamma_1} \phi, \nu_{\Gamma_1} E \rangle_{\mathsf{L}^2(\Gamma_1)}.$$

Here, the well-known scalar trace of the smooth function ϕ is given by $\sigma_\Gamma \phi = \phi|_\Gamma \in \mathsf{L}^2(\Gamma)$
with $\sigma_{\Gamma_0}\phi = \sigma_\Gamma \phi|_{\Gamma_0} = 0$ and $\sigma_{\Gamma_1}\phi = \sigma_\Gamma \phi|_{\Gamma_1} \in \mathsf{L}^2(\Gamma_1)$. Note that $\nu_{\Gamma_1} E$ is well defined as $\sigma_{\Gamma_1} \mathsf{H}^1_{\Gamma_0}(\Omega)$ is dense in $\mathsf{L}^2(\Gamma_1)$.

Remark 2.2 (L^2-traces.) Analogously to Definition and Remark 2.1 (i) and as

$$\tau^\times_{\Gamma_0} \widetilde{E} \cdot \tau_{\Gamma_0} \widetilde{H} = (n \times \widetilde{E}) \cdot (-n \times n \times \widetilde{H}) = (n \times n \times \widetilde{E}) \cdot (n \times \widetilde{H}) = -\tau_{\Gamma_0} \widetilde{E} \cdot \tau^\times_{\Gamma_0} \widetilde{H}$$

holds on Γ_0 for smooth vector fields $\widetilde{E}$, $\widetilde{H}$, we can define the twisted tangential trace $\tau^\times_{\Gamma_0} E \in \mathsf{L}^2(\Gamma_0)$ of a vector field $E \in \mathsf{H}(\mathrm{curl}, \Omega)$ as well by

$$\langle \mathrm{curl}\, \Phi, E \rangle_{\mathsf{L}^2(\Omega)} - \langle \Phi, \mathrm{curl}\, E \rangle_{\mathsf{L}^2(\Omega)} = -\langle \tau_{\Gamma_0} \Phi, \tau^\times_{\Gamma_0} E \rangle_{\mathsf{L}^2(\Gamma_0)}$$

for all vector fields $\Phi \in \mathsf{H}^1_{\Gamma_1}(\Omega)$.

3 Preliminaries

In [4, Theorem 5.5], see [3, Theorem 7.4] for more details and compare to [2], the following theorem about the existence of regular potentials for the rotation with homogeneous mixed boundary conditions has been shown.

Theorem 3.1 (regular potential for curl with homogeneous mixed boundary conditions)

$$\mathsf{H}_{\Gamma_1,0}(\mathrm{div}, \Omega) \cap \mathcal{H}_{\Gamma_0,\Gamma_1}(\Omega)^{\perp_{\mathsf{L}^2(\Omega)}} = \mathrm{curl}\, \mathsf{H}_{\Gamma_1}(\mathrm{curl}, \Omega) = \mathrm{curl}\, \mathsf{H}^1_{\Gamma_1}(\Omega)$$

holds together with a regular potential operator mapping $\mathrm{curl}\, \mathsf{H}_{\Gamma_1}(\mathrm{curl}, \Omega)$ to $\mathsf{H}^1_{\Gamma_1}(\Omega)$ continuously. In particular, the latter ranges are closed subspaces of $\mathsf{L}^2(\Omega)$.

Moreover, we need [4, Theorem 5.2]:

Theorem 3.2 (Helmholtz decompositions with homogeneous mixed boundary conditions) The ranges $\nabla \mathsf{H}^1_{\Gamma_0}(\Omega)$ and $\mathrm{curl}\, \mathsf{H}_{\Gamma_1}(\mathrm{curl}, \Omega)$ are closed subspaces of $\mathsf{L}^2(\Omega)$, and the $\mathsf{L}^2_\varepsilon(\Omega)$-orthogonal Helmholtz decompositions

$$\begin{aligned}\mathsf{L}^2_\varepsilon(\Omega) &= \nabla\,\mathsf{H}^1_{\Gamma_0}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \varepsilon^{-1}\mathsf{H}_{\Gamma_1,0}(\operatorname{div},\Omega)\\ &= \mathsf{H}_{\Gamma_0,0}(\operatorname{curl},\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \varepsilon^{-1}\operatorname{curl}\mathsf{H}_{\Gamma_1}(\operatorname{curl},\Omega)\\ &= \nabla\,\mathsf{H}^1_{\Gamma_0}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \varepsilon^{-1}\operatorname{curl}\mathsf{H}_{\Gamma_1}(\operatorname{curl},\Omega)\end{aligned}$$

hold (with continuous potential operators). Moreover, $\mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)$ has finite dimension.

Combining Theorem 3.1 and Theorem 3.2 shows immediately the following.

Corollary 3.3 *(regular Helmholtz decomposition with homogeneous mixed boundary conditions) The $\mathsf{L}^2_\varepsilon(\Omega)$-orthogonal regular Helmholtz decomposition*

$$L^2_\varepsilon(\Omega) = \nabla\,H^1_{\Gamma_0}(\Omega) \oplus_{L^2_\varepsilon(\Omega)} \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega) \oplus_{L^2_\varepsilon(\Omega)} \varepsilon^{-1}\operatorname{curl}H^1_{\Gamma_1}(\Omega)$$

holds (with continuous potential operators) and $\mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)$ has finite dimension. More precisely, any $E \in \mathsf{L}^2_\varepsilon(\Omega)$ may be $\mathsf{L}^2_\varepsilon(\Omega)$-orthogonally (and regularly) decomposed into

$$E = \nabla\,u_\nabla + E_{\mathcal{H}} + \varepsilon^{-1}\operatorname{curl}E_{\operatorname{curl}}$$

with $u_\nabla \in H^1_{\Gamma_0}(\Omega)$, $E_{\operatorname{curl}} \in H^1_{\Gamma_1}(\Omega)$, and $E_{\mathcal{H}} \in \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)$, and there exists a constant $c > 0$, independent of E, u_∇, $E_{\mathcal{H}}$, E_{curl}, such that

$$\begin{aligned}\|E_{\mathcal{H}}\|_{L^2_\varepsilon(\Omega)} &\le \|E\|_{L^2_\varepsilon(\Omega)},\\ c\|u_\nabla\|_{H^1_{\Gamma_0}(\Omega)} \le \|\nabla\,u_\nabla\|_{L^2_\varepsilon(\Omega)} &\le \|E\|_{L^2_\varepsilon(\Omega)},\\ c\|E_{\operatorname{curl}}\|_{H^1_{\Gamma_1}(\Omega)} \le \|\varepsilon^{-1}\operatorname{curl}E_{\operatorname{curl}}\|_{L^2_\varepsilon(\Omega)} &\le \|E\|_{L^2_\varepsilon(\Omega)}.\end{aligned}$$

4 Compact embeddings

Our main result reads as follows:

Theorem 4.1 (compact embedding for vector fields with inhomogeneous mixed boundary conditions)

$$\widehat{\mathsf{H}}_{\Gamma_0}(\operatorname{curl},\Omega) \cap \varepsilon^{-1}\widehat{\mathsf{H}}_{\Gamma_1}(\operatorname{div},\Omega) \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega).$$

Proof Let (E_ℓ) be a bounded sequence in $\widehat{\mathsf{H}}_{\Gamma_0}(\operatorname{curl},\Omega) \cap \varepsilon^{-1}\widehat{\mathsf{H}}_{\Gamma_1}(\operatorname{div},\Omega)$. By the Helmholtz decomposition in Corollary 3.3 we $\mathsf{L}^2_\varepsilon(\Omega)$-orthogonally and regularly decompose

$$E_\ell = \nabla\,u_{\nabla,\ell} + E_{\mathcal{H},\ell} + \varepsilon^{-1}\operatorname{curl}E_{\operatorname{curl},\ell}$$

with $u_{\nabla,\ell} \in \mathsf{H}^1_{\Gamma_0}(\Omega)$, $E_{\mathrm{curl},\ell} \in \mathsf{H}^1_{\Gamma_1}(\Omega)$, and $E_{\mathcal{H},\ell} \in \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)$, and there exists a constant $c > 0$ such that for all ℓ

$$\|u_{\nabla,\ell}\|_{\mathsf{H}^1_{\Gamma_0}(\Omega)} + \|E_{\mathcal{H},\ell}\|_{\mathsf{L}^2_\varepsilon(\Omega)} + \|E_{\mathrm{curl},\ell}\|_{\mathsf{H}^1_{\Gamma_1}(\Omega)} \leq c\|E_\ell\|_{\mathsf{L}^2_\varepsilon(\Omega)}.$$

As $\mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)$ is finite dimensional we may assume (after extracting a subsequence) that $E_{\mathcal{H},\ell}$ converges strongly in $\mathsf{L}^2_\varepsilon(\Omega)$. Since $\mathsf{H}^1(\Omega) \overset{\mathrm{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega)$ by Rellich's selection theorem, we may assume that also the regular potentials $u_{\nabla,\ell}$ and $E_{\mathrm{curl},\ell}$ converge strongly in $\mathsf{L}^2(\Omega)$. Moreover, $u_{\nabla,\ell}|_\Gamma$ and $E_{\mathrm{curl},\ell}|_\Gamma$ are bounded in $\mathsf{H}^{1/2}(\Gamma)$ by the (scalar) trace theorem, and thus we may assume by the compact embedding $\mathsf{H}^{1/2}(\Gamma) \overset{\mathrm{cpt}}{\hookrightarrow} \mathsf{L}^2(\Gamma)$ that $u_{\nabla,\ell}|_\Gamma$ and $E_{\mathrm{curl},\ell}|_\Gamma$ converge strongly in $\mathsf{L}^2(\Gamma)$. In particular, $u_{\nabla,\ell}|_{\Gamma_1}$ and $E_{\mathrm{curl},\ell}|_{\Gamma_0}$ converge strongly in $\mathsf{L}^2(\Gamma_1)$ and $\mathsf{L}^2(\Gamma_0)$, respectively. After all this successively taking subsequences we obtain (using $\mathsf{L}^2_\varepsilon(\Omega)$-orthogonality and the definition of the $\mathsf{L}^2(\Gamma_1)$-traces of $\nu_{\Gamma_1}\varepsilon E_\ell$ and the $\mathsf{L}^2(\Gamma_0)$-traces of $\tau_{\Gamma_0} E_\ell$ from Definition and Remark 2.1)

$$\begin{aligned}
&\|\nabla(u_{\nabla,\ell} - u_{\nabla,k})\|^2_{\mathsf{L}^2_\varepsilon(\Omega)}\\
&\quad= \langle \nabla(u_{\nabla,\ell} - u_{\nabla,k}), E_\ell - E_k\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}\\
&\quad= -\langle u_{\nabla,\ell} - u_{\nabla,k}, \operatorname{div}\varepsilon(E_\ell - E_k)\rangle_{\mathsf{L}^2(\Omega)} + \langle \sigma_{\Gamma_1}(u_{\nabla,\ell} - u_{\nabla,k}), \nu_{\Gamma_1}\varepsilon(E_\ell - E_k)\rangle_{\mathsf{L}^2(\Gamma_1)}\\
&\quad\leq c\|u_{\nabla,\ell} - u_{\nabla,k}\|_{\mathsf{L}^2(\Omega)} + c\|(u_{\nabla,\ell} - u_{\nabla,k})|_{\Gamma_1}\|_{\mathsf{L}^2(\Gamma_1)} \to 0
\end{aligned}$$

and

$$\begin{aligned}
&\|\varepsilon^{-1}\operatorname{curl}(E_{\mathrm{curl},\ell} - E_{\mathrm{curl},k})\|^2_{\mathsf{L}^2_\varepsilon(\Omega)}\\
&\quad= \langle \varepsilon^{-1}\operatorname{curl}(E_{\mathrm{curl},\ell} - E_{\mathrm{curl},k}), E_\ell - E_k\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}\\
&\quad= \langle E_{\mathrm{curl},\ell} - E_{\mathrm{curl},k}, \operatorname{curl}(E_\ell - E_k)\rangle_{\mathsf{L}^2(\Omega)} + \langle \tau^\times_{\Gamma_0}(E_{\mathrm{curl},\ell} - E_{\mathrm{curl},k}), \tau_{\Gamma_0}(E_\ell - E_k)\rangle_{\mathsf{L}^2(\Gamma_0)}\\
&\quad\leq c\|E_{\mathrm{curl},\ell} - E_{\mathrm{curl},k}\|_{\mathsf{L}^2(\Omega)} + c\|(E_{\mathrm{curl},\ell} - E_{\mathrm{curl},k})|_{\Gamma_0}\|_{\mathsf{L}^2(\Gamma_0)} \to 0.
\end{aligned}$$

Hence, (E_ℓ) contains a strongly $\mathsf{L}^2_\varepsilon(\Omega)$-convergent (and thus strongly $\mathsf{L}^2(\Omega)$-convergent) subsequence. □

Remark 4.2 (compact embedding for vector fields with inhomogeneous mixed boundary conditions.) Theorem 4.1 even holds for weaker boundary data. For this, let $0 \leq s < 1/2$. Taking into account the compact embedding $\mathsf{H}^{1/2}(\Gamma) \overset{\mathrm{cpt}}{\hookrightarrow} \mathsf{H}^s(\Gamma)$ and looking at the latter proof, we see that

$$\big\{E \in \mathsf{H}(\mathrm{curl},\Omega) \,:\, \tau_{\Gamma_0}E \in \mathsf{H}^{-s}(\Gamma_0)\big\} \cap \Big\{E \in \varepsilon^{-1}\mathsf{H}(\mathrm{div},\Omega) \,:\, \nu_{\Gamma_1}\varepsilon E \in \mathsf{H}^{-s}(\Gamma_1)\Big\} \overset{\mathrm{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega).$$

5 Applications

5.1 Friedrichs/poincaré type estimates

A first application is the following estimate:

Theorem 5.1 (Friedrichs/Poincaré type estimate for vector fields with inhomogeneous mixed boundary conditions) There exists a positive constant c such that for all vector fields E in $\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{curl}, \Omega) \cap \varepsilon^{-1}\widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div}, \Omega) \cap \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}}$ it holds

$$c\|E\|_{\mathsf{L}^2_\varepsilon(\Omega)} \leq \|\mathrm{curl}\, E\|_{\mathsf{L}^2(\Omega)} + \|\mathrm{div}\, \varepsilon E\|_{\mathsf{L}^2(\Omega)} + \|\tau_{\Gamma_0} E\|_{\mathsf{L}^2(\Gamma_0)} + \|\nu_{\Gamma_1} \varepsilon E\|_{\mathsf{L}^2(\Gamma_1)}.$$

Proof For a proof we use a standard compactness argument using Theorem 4.1. If the estimate was wrong, then there exists a sequence $(E_\ell) \in \widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{curl}, \Omega) \cap \varepsilon^{-1}\widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div}, \Omega) \cap \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}}$ with $\|E_\ell\|_{\mathsf{L}^2_\varepsilon(\Omega)} = 1$ and

$$\|\mathrm{curl}\, E_\ell\|_{\mathsf{L}^2(\Omega)} + \|\mathrm{div}\, \varepsilon E_\ell\|_{\mathsf{L}^2(\Omega)} + \|\tau_{\Gamma_0} E_\ell\|_{\mathsf{L}^2(\Gamma_0)} + \|\nu_{\Gamma_1} \varepsilon E_\ell\|_{\mathsf{L}^2(\Gamma_1)} \to 0.$$

Thus, by Theorem 4.1 (after extracting a subsequence)

$$E_\ell \to E \quad \text{in} \quad \widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{curl}, \Omega) \cap \varepsilon^{-1}\widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div}, \Omega) \cap \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}} \qquad \text{(strongly)}$$

and $\mathrm{curl}\, E = 0$ and $\mathrm{div}\, \varepsilon E = 0$ (by testing). Moreover, for all $\Phi \in \mathsf{C}^\infty_{\Gamma_1}(\Omega)$ and for all $\phi \in \mathsf{C}^\infty_{\Gamma_0}(\Omega)$

$$\begin{aligned}\langle \mathrm{curl}\, \Phi, E_\ell\rangle_{\mathsf{L}^2(\Omega)} - \langle \Phi, \mathrm{curl}\, E_\ell\rangle_{\mathsf{L}^2(\Omega)} &= \langle \tau^\times_{\Gamma_0} \Phi, \tau_{\Gamma_0} E_\ell\rangle_{\mathsf{L}^2(\Gamma_0)} \leq c\|\tau_{\Gamma_0} E_\ell\|_{\mathsf{L}^2(\Gamma_0)} \to 0,\\ \langle \nabla \phi, \varepsilon E_\ell\rangle_{\mathsf{L}^2(\Omega)} + \langle \phi, \mathrm{div}\, \varepsilon E_\ell\rangle_{\mathsf{L}^2(\Omega)} &= \langle \sigma_{\Gamma_1} \phi, \nu_{\Gamma_1} \varepsilon E_\ell\rangle_{\mathsf{L}^2(\Gamma_1)}\\ &\leq c\|\nu_{\Gamma_1} \varepsilon E_\ell\|_{\mathsf{L}^2(\Gamma_1)} \to 0,\end{aligned}$$

cf. Definition and Remark 2.1, implying

$$\langle \mathrm{curl}\, \Phi, E\rangle_{\mathsf{L}^2(\Omega)} = \langle \nabla \phi, \varepsilon E\rangle_{\mathsf{L}^2(\Omega)} = 0.$$

Hence, $E \in \mathsf{H}_{\Gamma_0,0}(\mathrm{curl}, \Omega) \cap \varepsilon^{-1}\mathsf{H}_{\Gamma_1,0}(\mathrm{div}, \Omega) = \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)$ by [4, Theorem 4.7] (weak and strong homogeneous boundary conditions coincide). This shows $E = 0$ as $E \perp_{\mathsf{L}^2_\varepsilon(\Omega)} \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)$, in contradiction to $1 = \|E_\ell\|_{\mathsf{L}^2_\varepsilon(\Omega)} \to \|E\|_{\mathsf{L}^2_\varepsilon(\Omega)} = 0$. □

Remark 5.2 (Friedrichs/Poincaré type estimate for vector fields with inhomogeneous mixed boundary conditions.) As in Remark 4.2 there are corresponding generalised Friedrichs/Poincaré type estimates for weaker boundary data, where the $\mathsf{L}^2(\Gamma_{0/1})$-spaces and norms are replaced by $\mathsf{H}^{-s}(\Gamma_{0/1})$-spaces and norms.

5.2 A div-curl lemma

Another immediate consequence is a div-curl-lemma.

Theorem 5.3 (div-curl lemma for vector fields with inhomogeneous mixed boundary conditions) Let (E_n) and (H_n) be bounded sequences in $\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$ and $\widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div}, \Omega)$, respectively. Then there exist $E \in \widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$ and $H \in \widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div}, \Omega)$ as well as subsequences, again denoted by (E_n) and (H_n), such that $E_n \rightharpoonup E$ in $\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$ and $H_n \rightharpoonup H$ in $\widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div}, \Omega)$ as well as

$$\langle E_n, H_n \rangle_{\mathsf{L}^2(\Omega)} \to \langle E, H \rangle_{\mathsf{L}^2(\Omega)}.$$

Proof We follow in closed lines the proof of [14, Theorem 3.1]. Let (E_n) and (H_n) be as stated. First, we pick subsequences, again denoted by (E_n) and (H_n), and E and H, such that $E_n \rightharpoonup E$ in $\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$ and $H_n \rightharpoonup H$ in $\widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div}, \Omega)$. In particular, $H_n \rightharpoonup H$ and $\mathrm{div}\, H_n \rightharpoonup \mathrm{div}\, H$ in $\mathsf{L}^2(\Omega)$ as well as

$$\nu_{\Gamma_1} H_n \rightharpoonup \nu_{\Gamma_1} H \quad \text{in} \quad \mathsf{L}^2(\Gamma_1). \tag{1}$$

To see (1), let $\nu_{\Gamma_1} H_n \rightharpoonup H_{\Gamma_1}$ in $\mathsf{L}^2(\Gamma_1)$. Since for all $\phi \in \mathsf{H}^1_{\Gamma_0}(\Omega)$

$$\begin{aligned} \langle \sigma_{\Gamma_1} \phi, H_{\Gamma_1} \rangle_{\mathsf{L}^2(\Gamma_1)} \leftarrow \langle \sigma_{\Gamma_1} \phi, \nu_{\Gamma_1} H_n \rangle_{\mathsf{L}^2(\Gamma_1)} &= \langle \nabla \phi, H_n \rangle_{\mathsf{L}^2(\Omega)} + \langle \phi, \mathrm{div}\, H_n \rangle_{\mathsf{L}^2(\Omega)} \\ &\to \langle \nabla \phi, H \rangle_{\mathsf{L}^2(\Omega)} + \langle \phi, \mathrm{div}\, H \rangle_{\mathsf{L}^2(\Omega)}, \end{aligned}$$

we get $H \in \widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div}, \Omega)$ and $\nu_{\Gamma_1} H = H_{\Gamma_1}$. Moreover, $\langle \sigma_{\Gamma_1} \phi, \nu_{\Gamma_1} H_n \rangle_{\mathsf{L}^2(\Gamma_1)} \to \langle \sigma_{\Gamma_1} \phi, \nu_{\Gamma_1} H \rangle_{\mathsf{L}^2(\Gamma_1)}$. As $\sigma_{\Gamma_1} \mathsf{H}^1_{\Gamma_0}(\Omega)$ is dense in $\mathsf{L}^2(\Gamma_1)$ and $\big(\langle \,\cdot\,, \nu_{\Gamma_1} H_n \rangle_{\mathsf{L}^2(\Gamma_1)}\big)$ is uniformly bounded with respect to n we obtain (1).

By Theorem 3.2 we have the orthogonal Helmholtz decomposition

$$\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{curl}, \Omega) \ni E_n = \nabla u_n + \widetilde{E}_n$$

with $u_n \in \mathsf{H}^1_{\Gamma_0}(\Omega)$ and $\widetilde{E}_n \in \widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{curl}, \Omega) \cap \mathsf{H}_{\Gamma_1,0}(\mathrm{div}, \Omega)$ as $\nabla \mathsf{H}^1_{\Gamma_0}(\Omega) \subset \mathsf{H}_{\Gamma_0,0}(\mathrm{curl}, \Omega) \subset \widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$. By orthogonality and the Friedrichs/Poincaré estimate, (u_n) is bounded in $\mathsf{H}^1_{\Gamma_0}(\Omega)$ and hence contains a strongly $\mathsf{L}^2(\Omega)$-convergent subsequence, again denoted by (u_n). (For $\Gamma_0 = \emptyset$ we may have to add a constant to each u_n.) Moreover, as $(u_n|_\Gamma)$ is bounded in $\mathsf{H}^{1/2}(\Gamma) \overset{\text{cpt}}{\hookrightarrow} \mathsf{L}^2(\Gamma)$ we may assume that $(u_n|_\Gamma)$ converges strongly in $\mathsf{L}^2(\Gamma)$. In particular, $(\sigma_{\Gamma_1} u_n) = (u_n|_{\Gamma_1})$ converges strongly in $\mathsf{L}^2(\Gamma_1)$. The sequence $(\widetilde{E}_n)$ is bounded in $\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{curl}, \Omega) \cap \mathsf{H}_{\Gamma_1,0}(\mathrm{div}, \Omega)$ by orthogonality and since $\mathrm{curl}\, \widetilde{E}_n = \mathrm{curl}\, E_n$ and $\tau_{\Gamma_0} \widetilde{E}_n = \tau_{\Gamma_0} E_n$. Theorem 4.1 yields a strongly $\mathsf{L}^2(\Omega)$-convergent subsequence, again denoted by $(\widetilde{E}_n)$. Hence, there exist $u \in \mathsf{H}^1_{\Gamma_0}(\Omega)$ and $\widetilde{E} \in \widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{curl}, \Omega) \cap \mathsf{H}_{\Gamma_1,0}(\mathrm{div}, \Omega)$ such that $u_n \rightharpoonup u$ in $\mathsf{H}^1_{\Gamma_0}(\Omega)$ and $u_n \to u$ in $\mathsf{L}^2(\Omega)$ and $\sigma_{\Gamma_1} u_n \to \sigma_{\Gamma_1} u$ in $\mathsf{L}^2(\Gamma_1)$ as well as $\widetilde{E}_n \rightharpoonup \widetilde{E}$ in $\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{curl}, \Omega) \cap \mathsf{H}_{\Gamma_1,0}(\mathrm{div}, \Omega)$ and

$\widetilde{E}_n \to \widetilde{E}$ in $\mathsf{L}^2(\Omega)$. Finally, we compute

$$\begin{aligned}\langle E_n, H_n\rangle_{\mathsf{L}^2(\Omega)} &= \langle \nabla u_n, H_n\rangle_{\mathsf{L}^2(\Omega)} + \langle \widetilde{E}_n, H_n\rangle_{\mathsf{L}^2(\Omega)}\\ &= -\langle u_n, \operatorname{div} H_n\rangle_{\mathsf{L}^2(\Omega)} + \langle \sigma_{\Gamma_1} u_n, \nu_{\Gamma_1} H_n\rangle_{\mathsf{L}^2(\Gamma_1)} + \langle \widetilde{E}_n, H_n\rangle_{\mathsf{L}^2(\Omega)}\\ &\to -\langle u, \operatorname{div} H\rangle_{\mathsf{L}^2(\Omega)} + \langle \sigma_{\Gamma_1} u, \nu_{\Gamma_1} H\rangle_{\mathsf{L}^2(\Gamma_1)} + \langle \widetilde{E}, H\rangle_{\mathsf{L}^2(\Omega)}\\ &= \langle \nabla u, \operatorname{div} H\rangle_{\mathsf{L}^2(\Omega)} + \langle \widetilde{E}, H\rangle_{\mathsf{L}^2(\Omega)} = \langle E, H\rangle_{\mathsf{L}^2(\Omega)},\end{aligned}$$

since indeed $E = \nabla u + \widetilde{E}$ holds by the weak convergence. □

Remark 5.4 (div-curl lemma for vector fields with inhomogeneous mixed boundary conditions.) As in Remark 4.2 and Remark 5.2 there are corresponding generalised div-curl lemmas for weaker boundary data, where the $\mathsf{L}^2(\Gamma_{0/1})$-spaces and norms are replaced by $\mathsf{H}^{-s}(\Gamma_{0/1})$-spaces and norms.

5.3 Maxwell's equations with mixed impedance type boundary conditions

Let ε, μ be admissible and time-independent matrix fields, and let $T, k \in \mathbb{R}_+$. In $I \times \Omega$ with $I := (0, T)$ we consider Maxwell's equations with mixed tangential and impedance boundary conditions

$$\begin{aligned}
\partial_t E - \varepsilon^{-1} \operatorname{curl} H &= F, && \text{(Ampère/Maxwell law)} && \text{(2a)}\\
\partial_t H + \mu^{-1} \operatorname{curl} E &= G, && \text{(Faraday/Maxwell law)} && \text{(2b)}\\
\operatorname{div} \varepsilon E &= \rho, && \text{(Gaußlaw)} && \text{(2c)}\\
\operatorname{div} \mu H &= 0, && \text{(Gauß law for magnetism)} && \text{(2d)}\\
\tau_{\Gamma_0} E &= 0, && \text{(perfect conductor bc)} && \text{(2e)}\\
\nu_{\Gamma_0} H &= f, && \text{(normal trace bc)} && \text{(2f)}\\
\tau_{\Gamma_1} E + k\tau^{\times}_{\Gamma_1} H &= 0, && \text{(impedance bc)} && \text{(2g)}\\
E(0) &= E_0, && \text{(electric initial value)} && \text{(2h)}\\
H(0) &= H_0. && \text{(magnetic initial value)} && \text{(2i)}
\end{aligned}$$

Here, F, G are time-dependent sources and E_0, H_0, ρ, and f are time-independent source terms. Note that the impedance boundary condition (also called Leontovich boundary condition) is of Robin type and that the impedance is given by $\lambda = 1/k = \sqrt{\varepsilon/\mu}$ if ε, μ are positive scalars.

Despite of other recent and very powerful approaches such as the concept of "evolutionary equations", see the pioneering work of Rainer Picard, e.g., [10, 20], one can use classical semigroup theory for solving the Maxwell system (2).

We will split the system (2) into two static systems and a dynamic system. For simplicity we set $\varepsilon = \mu = 1$ and $F = G = 0$. The static systems are

$$\begin{aligned} \operatorname{curl} E &= 0, & \operatorname{curl} H &= 0, && \text{(3a)}\\ \operatorname{div} E &= \rho, & \operatorname{div} H &= 0, && \text{(3b)}\\ \tau_{\Gamma_0} E &= 0, & \nu_{\Gamma_0} H &= f, && \text{(3c)}\\ \tau_{\Gamma_1} E &= -kg, & \tau_{\Gamma_1}^{\times} H &= g, && \text{(3d)} \end{aligned}$$

where g is any suitable tangential vector field in $\mathsf{L}^2(\Gamma_1)$. For simplicity we put $g = 0$, then these two systems are solvable by [2, Theorem 5.6]. However, the same result also gives conditions for which $g \neq 0$ this system is solvable. The dynamic system is

$$\begin{aligned} \partial_t E &= \operatorname{curl} H, && \text{(4a)}\\ \partial_t H &= -\operatorname{curl} E, && \text{(4b)}\\ \operatorname{div} E &= 0, && \text{(4c)}\\ \operatorname{div} H &= 0, && \text{(4d)}\\ \nu_{\Gamma_0} H &= 0, && \text{(4e)}\\ \tau_{\Gamma_0} E &= 0, && \text{(4f)}\\ \tau_{\Gamma_1} E + k\tau_{\Gamma_1}^{\times} H &= 0. && \text{(4g)} \end{aligned}$$

The initial conditions for the dynamic system are $E(0) = E_0 - E_s$ and $H(0) = H_0 - H_s$, where E_s and H_s are the solutions of the two static systems (3). We can write (4a) and (4b) as

$$\partial_t \begin{bmatrix} E \\ H \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{bmatrix}}_{=:\mathrm{A}_0} \begin{bmatrix} E \\ H \end{bmatrix},$$

and the boundary conditions (4f) and (4g) shall be covered by the domain of A_0:

$$\operatorname{dom} \mathrm{A}_0 := \left\{ (E, H) \in \widehat{\mathsf{H}}_{\Gamma}(\operatorname{curl}, \Omega) \times \widehat{\mathsf{H}}_{\Gamma_1}(\operatorname{curl}, \Omega) \;\middle|\; \tau_{\Gamma_0} E = 0,\ \tau_{\Gamma_1} E + k\tau_{\Gamma_1}^{\times} H = 0 \right\}.$$

Here, we did ignore the equations $\operatorname{div} E = 0$, $\operatorname{div} H = 0$ and $\nu_{\Gamma_0} H = 0$. However, A_0 is a generator of a C_0-semigroup by [22, Example 8.10] or [25, Section 5], where the input function is $u = 0$. (In these sources they regard boundary control systems and system nodes, respectively. One condition of those concepts is that the system with $u = 0$ is described by a generator of a C_0-semigroup). The following lemma provides a tool to show that the remaining conditions from (4) are also satisfied.

Lemma 5.5 *Let* $\mathrm{T}(\cdot)$ *be a* C_0*-semigroup on a Banach space* X*, and let* A *be its generator. Then every subspace* $V \supseteq \operatorname{ran} \mathrm{A}$ *is invariant under* $\mathrm{T}(\cdot)$*. Moreover,* $\mathrm{A}\big|_V$ *generates the strongly continuous semigroup* $\mathrm{T}_V(\cdot) := \mathrm{T}(\cdot)\big|_V$*, if* V *is additionally closed in* X*.*

Proof Let $t \geq 0$ and let $x \in V$. Then $\operatorname{ran} \mathrm{A} \ni \mathrm{A} \int_0^t \mathrm{T}(s)x \, \mathrm{d}s = \mathrm{T}(t)x - x$ and hence $\mathrm{T}(t)x \in V$. The remaining assertion follows from [6, Chapter II, Section 2.3]. □

Therefore, it is left to show that the remaining conditions establish a closed and invariant subspace under the semigroup T_0 generated by A_0 or contains $\operatorname{ran} \mathrm{A}_0$. Note that by Theorem 3.1

$$
\begin{aligned}
S &:= \{(E, H) \mid \operatorname{div} E = 0,\ \operatorname{div} H = 0,\ \nu_{\Gamma_0} H = 0\} \\
&= \mathsf{H}_0(\operatorname{div}, \Omega) \times \mathsf{H}_{\Gamma_0,0}(\operatorname{div}, \Omega) \\
&= \big(\operatorname{curl} \mathsf{H}(\operatorname{curl}, \Omega) \times \operatorname{curl} \mathsf{H}_{\Gamma_0}(\operatorname{curl}, \Omega)\big) \oplus \big(\mathcal{H}_{\Gamma,\emptyset}(\Omega) \times \mathcal{H}_{\Gamma_1,\Gamma_0}(\Omega)\big).
\end{aligned}
$$

This space is closed as the intersection of kernels of closed operators. Clearly, $\mathcal{H}_{\Gamma,\emptyset}(\Omega) \times \mathcal{H}_{\Gamma_1,\Gamma_0}(\Omega)$ is invariant under T_0, since every $(E, H) \in \mathcal{H}_{\Gamma,\emptyset}(\Omega) \times \mathcal{H}_{\Gamma_1,\Gamma_0}(\Omega)$ is a constant in time solution of the system (4), i.e.,

$$
\mathrm{T}_0(t) \begin{bmatrix} E \\ H \end{bmatrix} = \begin{bmatrix} E \\ H \end{bmatrix}.
$$

By

$$
\begin{aligned}
\operatorname{curl} \mathsf{H}(\operatorname{curl}, \Omega) \times \operatorname{curl} \mathsf{H}_{\Gamma_0}(\operatorname{curl}, \Omega) &= \begin{bmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{bmatrix} \big(\mathsf{H}_{\Gamma_0}(\operatorname{curl}, \Omega) \times \mathsf{H}(\operatorname{curl}, \Omega)\big) \\
&\supseteq \operatorname{ran} \mathrm{A}_0
\end{aligned}
$$

and Lemma 5.5 we have that also $\operatorname{curl} \mathsf{H}(\operatorname{curl}, \Omega) \times \operatorname{curl} \mathsf{H}_{\Gamma_0}(\operatorname{curl}, \Omega)$ is invariant under T_0. Hence, Lemma 5.5 and Theorem 4.1 imply the next theorem.

Theorem 5.6 $\mathrm{A} := \mathrm{A}_0\big|_S$ *is a generator of a* C_0*-semigroup and*

$$
\operatorname{dom} \mathrm{A} \subseteq \big(\hat{\mathsf{H}}\Gamma(\operatorname{curl}, \Omega) \cap \mathsf{H}(\operatorname{div}, \Omega)\big) \times \big(\hat{\mathsf{H}}\Gamma_1(\operatorname{curl}, \Omega) \cap \mathsf{H}\Gamma_0(\operatorname{div}, \Omega)\big) \overset{\mathrm{cpt}}{\hookrightarrow} \mathsf{L}2(\Omega).
$$

Consequently, every resolvent operator of A *is compact.*

If $\mathcal{H}_{\Gamma,\emptyset}(\Omega) = \{0\}$ and $\mathcal{H}_{\Gamma_1,\Gamma_0}(\Omega) = \{0\}$, then 0 is in the resolvent set of A and A^{-1} is compact. Alternatively, we can further restrict A to $\mathcal{H}_{\Gamma,\emptyset}(\Omega)^{\perp_{\mathsf{L}^2(\Omega)}} \times \mathcal{H}_{\Gamma_1,\Gamma_0}(\Omega)^{\perp_{\mathsf{L}^2(\Omega)}}$. This would also match our separation of static solutions and dynamic solutions, since solutions with initial condition in $\mathcal{H}_{\Gamma,\emptyset}(\Omega) \times \mathcal{H}_{\Gamma_1,\Gamma_0}(\Omega)$ are constant in time.

5.4 Wave equation with mixed impedance type boundary conditions

For the scalar wave equation the situation is even simpler since traces of $\mathsf{H}^1(\Omega)$-functions already belong to $\mathsf{L}^2(\Gamma)$, even to $\mathsf{H}^{1/2}(\Gamma)$. In $I \times \Omega$ we consider the wave equation in first order form (linear acoustics) with mixed scalar and impedance boundary conditions

$$\begin{aligned} \partial_t w - \operatorname{div} v &= 0, \\ \partial_t v - \nabla w &= 0, \\ \sigma_{\Gamma_0} w &= 0, \\ \sigma_{\Gamma_1} w + k \nu_{\Gamma_1} v &= 0, \\ w(0) &= w_0 \in \mathsf{L}^2(\Omega), \\ v(0) &= v_0 \in \mathsf{L}^2(\Omega). \end{aligned}$$

We write the system as

$$\partial_t \begin{bmatrix} w \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \operatorname{div} \\ \nabla & 0 \end{bmatrix}}_{=:\mathrm{A}_0} \begin{bmatrix} w \\ v \end{bmatrix}$$

with

$$\operatorname{dom} \mathrm{A}_0 := \left\{ (w, v) \in \mathsf{H}^1_{\Gamma_0}(\Omega) \times \widehat{\mathsf{H}}_{\Gamma_1}(\operatorname{div}, \Omega) \;\middle|\; \sigma_{\Gamma_1} w + k \nu_{\Gamma_1} = 0 \right\}.$$

As before, by [8, Theorem 4.4] or [22, Example 8.9], A_0 is a generator of C_0-semigroup. Again, we want to separate the static solutions from the dynamic system. The static solutions are given by $\ker \mathrm{A}_0$, which can be characterise by

$$\ker \mathrm{A}_0 = \{0\} \times \mathsf{H}_{\Gamma_1,0}(\operatorname{div}, \Omega),$$

where we assumed $\Gamma_0 \neq \emptyset$, otherwise the first component can also be constant and the second component would be in $\widehat{\mathsf{H}}_{\Gamma_1,0}(\operatorname{div}, \Omega)$. By Theorem 3.2, the orthogonal complement of $\ker \mathrm{A}_0$ is

$$S := \mathsf{L}^2(\Omega) \times \nabla \mathsf{H}^1_{\Gamma_0}(\Omega).$$

Note that S contains $\operatorname{ran} \mathrm{A}_0$ and is therefore (by Lemma 5.5) an invariant subspace under the semigroup generated by A_0. Moreover, note that $\nabla \mathsf{H}^1_{\Gamma_0}(\Omega) \subseteq \mathsf{H}_{\Gamma_0,0}(\operatorname{curl}, \Omega)$ and that S is closed. Hence, Lemma 5.5 and Theorem 4.1 imply the next theorem.

Theorem 5.7 $\mathrm{A} := \mathrm{A}_0 \big|_S$ *is a generator of* C_0*-semigroup and*

$$\operatorname{dom} \mathrm{A} \subseteq \mathsf{H}1\Gamma_0(\Omega) \times \left(\hat{\mathsf{H}}\Gamma_1(\operatorname{div}, \Omega) \cap \mathsf{H}\Gamma_0, 0(\operatorname{curl}, \Omega) \right) \overset{\mathrm{cpt}}{\hookrightarrow} \mathsf{L}2(\Omega).$$

Consequently, every resolvent operator of A *is compact.*

Alternatively, we can also regard the classical formulation of the wave equation and see that it is necessary for the second component in our formulation to be in $\nabla \mathsf{H}^1_{\Gamma_0}(\Omega)$, if we want the solutions to correspond.

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Declarations

Conflict of interest This work does not have any conflict of interest.

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