ANALYSIS OF COUPLED MAXWELL-CABLE PROBLEMS

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Abstract. Building on the recently published work [CGRS25], which introduces a model for the interaction between electromagnetic fields and radiating (possibly curved) cables, we analyze the qualitative properties of the resulting dynamical system. The model features inputs and outputs given by the currents and voltages at the cable ends, while the state comprises the corresponding distributions along the cables and the electromagnetic fields in the surrounding domain. We show that the autonomous dynamics (i.e., with zero input) generate a strongly continuous semigroup and establish sufficient conditions for well-posedness, meaning continuous dependence of the state and output trajectories on the inputs and initial conditions.

1. Introduction

The present work builds on the mathematical model introduced in [CGRS25] for describing the interaction between electromagnetic fields and radiating cables. In this framework, the electromagnetic field is governed by Maxwell's equations, whereas the cables (referred to synonymously as *transmission lines* throughout this work) are described by the telegrapher's equations. The coupling between these subsystems is realized through interface conditions that link the cable currents to the magnetic field strength and the cable voltages to the electric field strength.

A particular challenge of this setting lies in its mixed-dimensional character. The currents and voltages along the cables are functions defined on one-dimensional intervals, while the coupling interface consists of the cable surfaces, that is, two-dimensional submanifolds of $\mathbb{R}^3$. The electromagnetic field dynamics, in turn, evolve in the three-dimensional exterior domain surrounding the cables. Altogether, the resulting model comprises coupled telegrapher's and Maxwell's equations.

A similar mixed-dimensional coupling was used in [JSE23] for heat exchange, albeit without examining well-posedness.

There exists a vast body of literature in electrical engineering on this type of coupled problems (see, e.g., [RRPR02, LWK+17, Rac12, LNT88, PA81, APG80]). However, the simultaneous consideration of curved cables and a rigorous mathematical analysis, including solvability and well-posedness, appears to be missing.

In this article, we address these aspects from the viewpoint of infinite-dimensional linear systems theory and the theory of port-Hamiltonian systems. We consider k cables with circular cross-sections and possibly curved geometries, which interact with the surrounding electromagnetic field. The voltage and current distributions along the cables are extended to their lateral surfaces, forming a two-dimensional interface where these quantities serve as tangential boundary data for the electric and magnetic field intensities governed by Maxwell's equations in the exterior domain. The inputs and outputs of the overall system are given by the boundary values of the telegrapher's equations, i.e., by the voltages and currents at the cable ends.

Date: October 23, 2025.

2020 *Mathematics Subject Classification.* 35Q61, 37L05, 78A25, 93B28, 93C25.

Key words and phrases. Maxwell's equations, transmission line, system nodes, infinite-dimensional linear systems, port-Hamiltonian systems, well-posed linear systems.

Since the state of the overall system consists of functions of spatial variables, the state space is infinite-dimensional, whereas the input and output spaces are finite-dimensional, determined by linear combinations of the boundary voltages and currents. We prove that the free (autonomous) dynamics generate a strongly continuous semigroup. By employing the theory of system nodes [Sta05], we derive conditions on the input and the initial state that ensure existence of solutions. Finally, we provide conditions on the input configuration that guarantee well-posedness, i.e., continuous dependence of the state and output on the input and the initial state.

This article is organized as follows. After introducing the notation in the remainder of this section, we present the mathematical model in Section 2. In Section 3, we derive several key properties of the coupling conditions that are essential for the subsequent analysis. Section 4 reformulates the coupled problem in an operator-theoretic framework and characterizes all boundary conditions that render the system dissipative. In Section 5, we focus on the input–output behavior of the coupled field–cable system. After introducing a system-node formulation, we establish criteria for well-posedness and demonstrate that the resulting model fits naturally into the port-Hamiltonian framework.

Notation and convention. Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Hilbert spaces. The Cartesian product of $\mathcal{X}$ and $\mathcal{Y}$ is commonly represented as $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]_\times$, and the extension to more than two spaces is straightforward. We denote the norm in $\mathcal{X}$ as $\|\cdot\|_\mathcal{X}$ and the identity mapping in $\mathcal{X}$ as $\mathrm{id}_\mathcal{X}$. Similarly, we use id_n for the identity mapping on $\mathbb{C}^n$.

We omit the subscripts indicating the space when the context is clear. Further, if not stated else, a Hilbert space is canonically identified with its anti-dual.

The space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\mathcal{L}_\mathrm{b}(\mathcal{X},\mathcal{Y})$. As usual, we abbreviate $\mathcal{L}_\mathrm{b}(\mathcal{X}) \coloneqq \mathcal{L}_\mathrm{b}(\mathcal{X},\mathcal{X})$. The domain $\mathrm{dom}(A)$ of a possibly unbounded linear operator $A\colon \mathrm{dom}(A) \subseteq \mathcal{X} \to \mathcal{Y}$ is typically equipped with the graph norm $\|x\|_{\mathrm{dom}(A)} \coloneqq \left(\|x\|_\mathcal{X}^2 + \|Ax\|_\mathcal{Y}^2\right)^{1/2}$. By writing $A \subset B$ for two operators, we mean that A is a restriction of B, and $\overline{A}$ stands for the closure of a closable linear operator A. The adjoint of a densely defined linear operator $A\colon \mathrm{dom}(A) \subseteq \mathcal{X} \to \mathcal{Y}$ is $A^*\colon \mathrm{dom}(A^*) \subseteq \mathcal{Y} \to \mathcal{X}$ with

$$\mathrm{dom}(A^*) = \{y \in \mathcal{Y} \,|\, \exists\, z \in \mathcal{X} \text{ s.t. } \forall\, x \in \mathrm{dom}(A) : \langle y, Ax\rangle_\mathcal{Y} = \langle z, x\rangle_\mathcal{X}\}.$$

The vector $z \in \mathcal{X}$ in the above set is uniquely determined by $y \in \mathrm{dom}(A^*)$, and we set $A^*y = z$. Note that we identify $\mathbb{C}^{n\times m} \cong \mathcal{L}_\mathrm{b}(\mathbb{C}^m, \mathbb{C}^n)$. Together with the fact that $\mathbb{C}^n$ and $\mathbb{C}^m$ are equipped with the Euclidean inner product, this means that $A^* \in \mathbb{C}^{n\times m}$ is the conjugate transpose of $A \in \mathbb{C}^{m\times n}$. Likewise, x^* is the conjugate transpose of $x \in \mathbb{C}^n \cong \mathbb{C}^{n\times 1}$, such that the inner product in $\mathbb{C}^n$ reads

$$\langle x, y\rangle_{\mathbb{C}^n} = y^*x.$$

For $P \in \mathbb{C}^{n\times n}$, we write $P > 0$ ($P \geq 0$), if $P = P^*$ is positive (semi-)definite. Likewise, $P < 0$ ($P \leq 0$) means that $P = P^*$ is negative (semi-)definite. Further, $A^\dagger \in \mathbb{C}^{n\times m}$ denotes the Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$.

We use the notation of the widely used textbook [AF03] by Adams et al. for Lebesgue and Sobolev spaces. For function spaces with values in a Hilbert space $\mathcal{X}$, we indicate the additional mark "$;\mathcal{X}$" after writing the domain. For instance, the Lebesgue space of p-integrable $\mathcal{X}$-valued functions on the domain Ω is $\mathrm{L}^p(\Omega;\mathcal{X})$. Note that, throughout this article, integration of $\mathcal{X}$-valued functions always has to be understood in the Bochner sense [DU77].

2. The model

We now present the details of our model. To this end, we describe the assumptions on the spatial geometry of the problem, the modeling of the cables including the assumptions on the input–output configuration, the modeling of the electromagnetic field, and the coupling between the cable and the field.

Throughout the entire article, k denotes the number of cables.

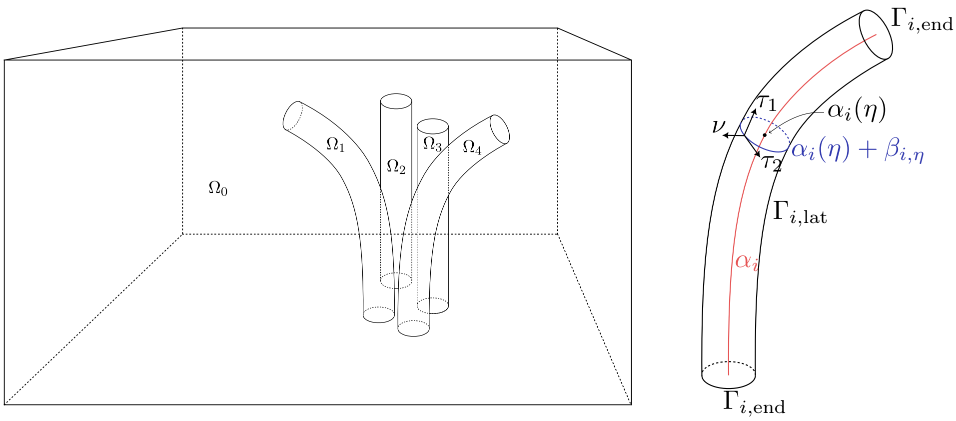
2.1. **The geometry.** We assume that the electromagnetic field evolves within a domain $\Omega \subseteq \mathbb{R}^3$ structured as

$$\Omega = \Omega_0 \setminus \bigcup_{i=1}^{k} \overline{\Omega_i},$$

where $\Omega_0 \subseteq \mathbb{R}^3$ Lipschitz domain, and the sets $\Omega_1, \dots, \Omega_k \subseteq \mathbb{R}^3$ fulfill

$$\overline{\Omega_i} \subseteq \Omega_0, \quad i = 1, \dots, k,$$
$$\overline{\Omega_i} \cap \overline{\Omega_j} = \emptyset, \quad i, j = 1, \dots, k \text{ with } i \neq j$$

All physical processes under consideration take place within Ω_0, which therefore represents the region enclosing the entire field–cable system. This region will be referred to as the computational domain. No boundedness of Ω_0 is required, and in particular, choosing $\Omega_0 = \mathbb{R}^3$ is admissible. The subdomains $\Omega_1, \dots, \Omega_k$ denote the spatial regions occupied by the individual cables, as illustrated in Figure 1a. Each of them is assumed to have a tubular geometry, as will be specified below.



(A) Spatial domain with cables (B) Cable parameterization

FIGURE 1. Cable geometry

The cables are allowed to be bent, and we assume that each cable has a circle-shaped cross-sectional area of constant radius. Denote the radius of this cross-sectional area of the ith cable by r_i, and let l_i be its length. The center curve (i.e., the curve whose trace is consisting of the centers of the cross-sectional circles) is denoted by $\alpha_i \colon [0,1] \to \mathbb{R}^3$, see Figure 1b. We assume the center curve to be twice continuously differentiable with constant infinitesimal arc length l_i, and curvature is bounded by $\frac{1}{r_i}$, i.e.,

$$\|\alpha_i'(\eta)\| = l_i \quad \text{and} \quad \|\alpha''(\eta)\| < \frac{l_i^2}{r_i} \quad \text{for all} \quad \eta \in [0,1].$$

It has been shown in [CGRS25, Lem. C.1] that there exist $\kappa_{i1}, \kappa_{i2} \in \mathrm{C}^1([0,1], \mathbb{R}^3)$, such that, for all $\eta \in [0,1]$, $\left(\frac{1}{l_i}\alpha'(\eta), \kappa_{i1}(\eta), \kappa_{i2}(\eta)\right)$ is a positively oriented orthonormal basis of $\mathbb{R}^3$ (that is, it forms a rotation matrix for all $\eta \in [0,1]$).

The shape of the ith cable shape is now expressed by

$$\Omega_i = \big\{\alpha_i(\eta) + \delta\beta_{i\eta}(\theta) \,\big|\, (\delta,\eta,\theta) \in [0,1]^2 \times (-\pi,\pi]\big\},$$
$$\text{with} \quad \beta_{i\eta}(\theta) = r_i\big(\kappa_{i1}(\eta)\sin(\theta) + \kappa_{i2}(\eta)\cos(\theta)\big).$$

and its boundary is consisting of the disjoint union $\partial\Omega_i = \Gamma_{i,\mathrm{lat}} \mathbin{\dot\cup} \Gamma_{i,\mathrm{end}}$, where $\Gamma_{i,\mathrm{end}}$ is the union of cross-sectional areas at the two ends of the ith cable and $\Gamma_{i,\mathrm{lat}}$ is the lateral surface of the ith cable. That is, for $\beta_{i\eta}(\theta)$ as above,

$$\begin{aligned}\Gamma_{i,\mathrm{end}} &= \big\{\alpha_i(\eta) + \delta\beta_{i\eta}(\theta) \,\big|\, (\delta,\eta,\theta) \in \{0,1\} \times [0,1) \times (-\pi,\pi]\big\},\\ \Gamma_{i,\mathrm{lat}} &= \big\{\alpha_i(\eta) + \beta_{i\eta}(\theta) \,\big|\, (\eta,\theta) \in (0,1) \times (-\pi,\pi]\big\}.\end{aligned}$$

The lateral boundary $\Gamma_{i,\mathrm{lat}}$ of the ith cable is now parameterised by

$$\Phi_i\colon \begin{cases} [0,1]\times(-\pi,\pi] & \to \quad \mathbb{R}^3,\\ \qquad (\eta,\theta) & \mapsto \quad \alpha_i(\eta) + \beta_{i\eta}(\theta) \end{cases} \qquad\qquad (1)$$
$$\text{with} \quad \beta_{i\eta}(\theta) = r_i\big(\kappa_{i1}(\eta)\sin(\theta) + \kappa_{i2}(\eta)\cos(\theta)\big).$$

The requirement that the curvature of the cable's profile curve is strictly limited by $\frac{1}{r_i}$ (i.e., $\|\alpha''(\eta)\| \leq \frac{l_i^2}{r_i}$) ensures that the parametrization (1) of the lateral surface is essentially injective.

2.2. **The transmission lines.** The k transmission lines are described by the *telegrapher's equations*, extended by external current inputs and electric field outputs. The internal variables of the model are the $\mathbb{C}^k$-valued functions

$\boldsymbol{\psi}$: magnetic flux,
$\boldsymbol{q}$: electric charge,

where each component corresponds to the flux (respectively, the charge) associated with one of the transmission lines. Both quantities depend on time t and on the spatial coordinate $\eta \in [0,1]$. The system is driven by an external current input $\boldsymbol{I}_{\mathrm{ext}} : \mathbb{R}_{\geq 0} \times [0,1] \to \mathbb{C}^k$, and the corresponding external electric field intensity $\boldsymbol{E}_{\mathrm{ext}} : \mathbb{R}_{\geq 0} \times [0,1] \to \mathbb{C}^k$ is taken as the output. These quantities, $\boldsymbol{I}_{\mathrm{ext}}$ and $\boldsymbol{E}_{\mathrm{ext}}$, will later serve as coupling variables between the transmission lines and the electromagnetic field. The transmission line model is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{\psi}(t,\eta)\\ \boldsymbol{q}(t,\eta)\end{pmatrix} &= \begin{bmatrix}-\boldsymbol{R}(\eta) & -\frac{\partial}{\partial\eta}\\ -\frac{\partial}{\partial\eta} & -\boldsymbol{G}(\eta)\end{bmatrix}\begin{pmatrix}\boldsymbol{L}(\eta)^{-1}\boldsymbol{\psi}(t,\eta)\\ \boldsymbol{C}(\eta)^{-1}\boldsymbol{q}(t,\eta)\end{pmatrix} + \begin{bmatrix}0\\ -\frac{\partial}{\partial\eta}\end{bmatrix}\boldsymbol{I}_{\mathrm{ext}}(t,\eta),\\ \boldsymbol{E}_{\mathrm{ext}}(t,\eta) &= \begin{bmatrix}0 & \frac{\partial}{\partial\eta}\end{bmatrix}\begin{pmatrix}\boldsymbol{L}(\eta)^{-1}\boldsymbol{\psi}(t,\eta)\\ \boldsymbol{C}(\eta)^{-1}\boldsymbol{q}(t,\eta)\end{pmatrix},\end{aligned} \qquad (2)$$

where the mappings $\boldsymbol{C},\boldsymbol{G},\boldsymbol{L},\boldsymbol{R}\colon [0,1] \to \mathbb{C}^{k\times k}$ denote, respectively, the transverse capacitance and conductance matrices, and the longitudinal inductance and resistance matrices. For any $t > 0$, the voltage and total current along the transmission lines are represented by the functions $\boldsymbol{V},\boldsymbol{I}_{\mathrm{tot}}\colon \mathbb{R}_{\geq 0} \times [0,1] \to \mathbb{C}^k$, defined as

$$\boldsymbol{V}(t,\eta) \coloneqq \boldsymbol{C}(\eta)^{-1}\boldsymbol{q}(t,\eta), \qquad \boldsymbol{I}_{\mathrm{tot}}(t,\eta) \coloneqq \boldsymbol{L}(\eta)^{-1}\boldsymbol{\psi}(t,\eta) + \boldsymbol{I}_{\mathrm{ext}}(t,\eta). \qquad (3)$$

Our assumptions concerning the parameters align with those presented in [JZ12].

Assumption 2.1 (Transmission lines - parameters)**.** $k \in \mathbb{N}$, $\boldsymbol{C}$, $\boldsymbol{L}$, $\boldsymbol{R}$, $\boldsymbol{G} \in \mathrm{L}^\infty([0,1];\mathbb{C}^{k\times k})$ with $\boldsymbol{C}^{-1},\boldsymbol{L}^{-1} \in \mathrm{L}^\infty([0,1];\mathbb{C}^{k\times k})$ and

$$\boldsymbol{C}(\eta) > 0,\ \boldsymbol{L}(\eta) > 0,\ \boldsymbol{R}(\eta) + \boldsymbol{R}(\eta)^* \geq 0,\ \boldsymbol{G}(\eta) + \boldsymbol{G}(\eta)^* \geq 0 \quad \text{for a.e.} \quad \eta \in [0,1].$$

The system is provided with an initial condition $\boldsymbol{q}(0,\eta) = \boldsymbol{q}_0(\eta)$, $\boldsymbol{\psi}(0,\eta) = \boldsymbol{\psi}_0(\eta)$ for some given $\boldsymbol{q}_0,\boldsymbol{\psi}_0 : [0,1] \to \mathbb{C}^k$. We further impose boundary conditions for the voltage and current along the transmission line, as given in (3). For some $m \leq 2k$, with $W_{B,\mathrm{inp}} \in \mathbb{C}^{m\times 4k}$ and $W_{B,0} \in \mathbb{C}^{(2k-m)\times 4k}$, these are given by

$$u(t) = W_{B,\mathrm{inp}} \begin{pmatrix} \boldsymbol{V}(t,0) \\ \boldsymbol{I}(t,0) \\ \boldsymbol{V}(t,1) \\ -\boldsymbol{I}(t,1) \end{pmatrix}, \quad 0 = W_{B,0} \begin{pmatrix} \boldsymbol{V}(t,0) \\ \boldsymbol{I}(t,0) \\ \boldsymbol{V}(t,1) \\ -\boldsymbol{I}(t,1) \end{pmatrix}, \tag{4}$$

where $u\colon \mathbb{R}_{\geq 0} \to \mathbb{C}^m$ represents the input of the system. The negative sign in the current at $\eta = 1$ reflects that, unlike at $\eta = 0$, the directions of current and voltage are opposite.

Assumption 2.2 (Transmission line - input configuration)**.** The matrix $W_B := [W_{B,\mathrm{inp}}^*, W_{B,0}^*]^* \in \mathbb{C}^{2k\times 4k}$ has full row rank, and

$$W_B \begin{bmatrix} 0 & \mathrm{id}_{2k} \\ \mathrm{id}_{2k} & 0 \end{bmatrix} W_B^* \geq 0. \tag{5}$$

Physical interpretations of the above assumption are discussed in [CGRS25], for example in the context of parallel or serial interconnections with ohmic resistors.
Our system is moreover equipped with an $\mathbb{C}^p$-valued output y of the form

$$y(t) = W_{C,\mathrm{out}} \begin{pmatrix} \boldsymbol{V}(t,0) \\ \boldsymbol{I}(t,0) \\ \boldsymbol{V}(t,1) \\ -\boldsymbol{I}(t,1) \end{pmatrix}, \tag{6}$$

for some $W_{C,\mathrm{out}} \in \mathbb{C}^{p\times 4m}$. A special role is played by so-called *co-located outputs.*

Definition 2.3. Assume that $W_{B,\mathrm{inp}} \in \mathbb{C}^{m\times 4k}$, $W_{B,0} \in \mathbb{C}^{(2k-m)\times 4k}$, $m \leq 2k$, $W_B := [W_{B,\mathrm{inp}}^*, W_{B,0}^*]^* \in \mathbb{C}^{2m\times 4m}$ fulfill Assumption 2.2. Then an output (6) is called *co-located to u as in* (3)*,* (4)*, if $W_{C,\mathrm{out}} \in \mathbb{C}^{m\times 4k}$ has the form*

$$W_{C,\mathrm{out}} = [\mathrm{id}_m,\, 0_{m\times(2k-m)}]\, W_C \tag{7}$$

for some $W_C \in \mathbb{C}^{2k\times 4k}$ with the property that $[W_B^*, W_C^*] \in \mathbb{C}^{4k\times 4k}$ with

$$\begin{bmatrix} 0 & \mathrm{id}_{2k} \\ \mathrm{id}_{2k} & 0 \end{bmatrix} - \begin{bmatrix} W_B \\ W_C \end{bmatrix}^* \begin{bmatrix} 0 & \mathrm{id}_{2k} \\ \mathrm{id}_{2k} & 0 \end{bmatrix} \begin{bmatrix} W_B \\ W_C \end{bmatrix} \leq 0. \tag{8}$$

The existence of co-located outputs is established in [CGRS25, Prop. 2.7], where a subsequent discussion also provides their physical interpretation. It is further shown there that such outputs give rise to a system satisfying an energy balance, where the inner product of the input and output corresponds to the power supplied to the system. Our transmission line model can now be viewed from a systems-theoretic perspective as a system featuring two classes of inputs and outputs: the external input together with the distributed current along the cable, and the external output accompanied by the corresponding distributed voltages.

2.3. **The electromagnetic field.** For the domain $\Omega \subset \mathbb{R}^3$ introduced in Section 2.1, the evolution of the electromagnetic field in the linear case is governed by Maxwell's equations

$$\frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{B}(t,\xi) \\ \boldsymbol{D}(t,\xi) \end{pmatrix} = \begin{bmatrix} 0 & -\mathrm{rot} \\ \mathrm{rot} & -\boldsymbol{\sigma}(\xi) \end{bmatrix} \begin{pmatrix} \boldsymbol{\mu}(\xi)^{-1}\boldsymbol{B}(t,\xi) \\ \boldsymbol{\epsilon}(\xi)^{-1}\boldsymbol{D}(t,\xi) \end{pmatrix} \tag{9}$$

for the $\mathbb{C}^3$-valued physical quantities

$\boldsymbol{B}$: magnetic flux density,
$\boldsymbol{D}$: electric flux density.

Here, $\boldsymbol{\mu}\colon \Omega \to \mathbb{C}^{3\times 3}$ denotes the magnetic permeability, $\boldsymbol{\epsilon}\colon \Omega \to \mathbb{C}^{3\times 3}$ the electric permittivity, and $\boldsymbol{\sigma}\colon \Omega \to \mathbb{C}^{3\times 3}$ the electric conductivity. The quantities

$$\boldsymbol{H}(t,\xi) \coloneqq \boldsymbol{\mu}(\xi)^{-1}\boldsymbol{B}(t,\xi), \quad \boldsymbol{E}(t,\xi) \coloneqq \boldsymbol{\epsilon}(\xi)^{-1}\boldsymbol{D}(t,\xi)$$

are referred to as the magnetic and electric field intensities, respectively. Typically, Maxwell's equations are complemented by the constraints

$$\operatorname{div}\boldsymbol{B}(t,\xi) = 0, \qquad \operatorname{div}\boldsymbol{D}(t,\xi) = \boldsymbol{\rho}(t,\xi),$$

where $\boldsymbol{\rho}(t,\cdot)\colon \Omega \to \mathbb{C}$ is a scalar field representing the charge density at time t. Since $\operatorname{div}\operatorname{rot} = 0$, these divergence relations are preserved in time and can therefore be imposed through the initial conditions; consequently, they will be omitted.

Our assumptions on the corresponding parameters are as follows.

Assumption 2.4 (Maxwell's equations - parameters)**.** We assume that $\boldsymbol{\epsilon}, \boldsymbol{\mu}, \boldsymbol{\sigma} \in \mathrm{L}^\infty(\Omega;\mathbb{C}^{3\times 3})$ with $\boldsymbol{\epsilon}^{-1}, \boldsymbol{\mu}^{-1} \in \mathrm{L}^\infty(\Omega;\mathbb{C}^{3\times 3})$ and

$$\boldsymbol{\epsilon}(\xi) > 0,\ \boldsymbol{\mu}(\xi) > 0,\ \boldsymbol{\sigma}(\xi) + \boldsymbol{\sigma}(\xi)^* \geq 0 \quad \text{for almost all} \quad \xi \in \Omega.$$

The system is provided with an initial condition $\boldsymbol{B}(0) = \boldsymbol{B}_0$, $\boldsymbol{D}(0) = \boldsymbol{D}_0$ for some given $\boldsymbol{B}_0, \boldsymbol{D}_0\colon \Omega \to \mathbb{C}^3$. To ensure a physically complete description, it is necessary to define appropriate boundary conditions.

The geometry of the domain Ω implies that its boundary $\partial\Omega$ is the disjoint union of the sub-boundaries $\partial\Omega_0, \partial\Omega_1, \ldots, \partial\Omega_k$. Each cable domain Ω_i has a boundary composed of its lateral surface $\Gamma_{i,\mathrm{lat}}$ and its two end surfaces $\Gamma_{i,\mathrm{end}}$, $i = 1, \ldots, k$. Consequently,

$$\partial\Omega = \partial\Omega_0 \mathbin{\dot\cup} \Gamma_{\mathrm{end}} \mathbin{\dot\cup} \Gamma_{\mathrm{lat}}, \qquad \Gamma_{\mathrm{end}} \coloneqq \dot{\bigcup_{i=1}^{k}} \Gamma_{i,\mathrm{end}}, \quad \Gamma_{\mathrm{lat}} \coloneqq \dot{\bigcup_{i=1}^{k}} \Gamma_{i,\mathrm{lat}}.$$

We next specify boundary conditions on $\partial\Omega_0$, Γ_{end}, and Γ_{lat}. To this end, let $\nu(\xi) \in \mathbb{R}^3$ denote the outward unit normal, which exists for almost every $\xi \in \partial\Omega$ (since Ω is a Lipschitz domain). Hence, $\nu \in \mathrm{L}^\infty(\partial\Omega;\mathbb{C}^3)$. For almost every $\xi \in \partial\Omega$, we define the orthogonal projection on the tangent space, which reads

$$\pi_\tau(\xi)\colon \mathbb{C}^3 \to \mathbb{C}^3, \qquad w \mapsto (\nu(\xi) \times w) \times \nu(\xi).$$

2.3.1. *Boundary conditions at the computational domain.* In the following analysis, we impose the boundary condition

$$\pi_\tau(\cdot)\big(\boldsymbol{\epsilon}(\cdot)^{-1}\boldsymbol{D}(t,\cdot)\big)\big|_{\partial\Omega_0} = 0, \qquad t \geq 0, \tag{10}$$

which corresponds to perfect electrical insulation outside Ω_0. Alternatively, superconductivity in the exterior region can be modeled by

$$\nu(\cdot) \times \big(\boldsymbol{\mu}(\cdot)^{-1}\boldsymbol{B}(t,\cdot)\big)\big|_{\partial\Omega_0} = 0, \qquad t \geq 0,$$

or by employing the *Leontovich boundary conditions* [Leo44], which relate the tangential components of the electric and magnetic fields. An analogous analysis could be carried out for these cases, but this is not pursued here.

2.3.2. *Boundary conditions at the cover surfaces of the cable.* We assume that the electric field is polarized normally to the cable end surfaces, which leads to

$$\pi_\tau(\cdot)(\boldsymbol{\epsilon}(\cdot)^{-1}\boldsymbol{D}(t,\cdot))\big|_{\Gamma_{\mathrm{end}}} = 0, \qquad t \geq 0. \tag{11}$$

2.4. **Coupling - transmission line and lateral cable surfaces.** The lateral cable surfaces serve as the coupling interface between the external electromagnetic field, governed by Maxwell's equations (9), and the transmission line, described by the telegrapher's equations (2).
The coupling between the transmission-line current and the magnetic field intensity follows from Ampère's law, relating the line integral of the magnetic field along a closed contour encircling the conductor to the enclosed current. For $i = 1, \dots, k$,

$$\forall \eta \in [0,1] \ : \ \oint_{\alpha_i(\eta)+\beta_{i\eta}} \underbrace{\boldsymbol{\mu}(\xi)^{-1}\boldsymbol{B}(t,\xi)}_{=\boldsymbol{H}(t,\xi)} \cdot \mathrm{d}\boldsymbol{s}(\xi) = -\boldsymbol{I}_{i,\mathrm{ext}}(t,\eta),$$

where $\boldsymbol{I}_{i,\mathrm{ext}}$ is the ith component of $\boldsymbol{I}_{\mathrm{ext}}(t,\eta)$. Now using that the above integral only incorporated the tangential component of the magnetic field, we see that Ampère's law is equivalent to, for $i = 1, \dots, k$,

$$\forall \eta \in [0,1] \ : \ \oint_{\alpha_i(\eta)+\beta_{i\eta}} \big(\nu(\xi) \times \big(\boldsymbol{\mu}(\xi)^{-1}\boldsymbol{B}(t,\xi)\big)\big) \times \nu(\xi) \cdot \mathrm{d}\boldsymbol{s}(\xi) = -\boldsymbol{I}_{i,\mathrm{ext}}(t,\eta). \quad (12)$$

The coupling between the external voltage variable $\boldsymbol{E}_{\mathrm{ext}}$ of the transmission line and the electric field on the cable surface relies on the polarization assumptions that the electric field is tangentially aligned with the cable axis so that its line integral along a longitudinal path equals the voltage drop along the line, and that it vanishes in the perpendicular direction. According to the results in [CGRS25, Sec. 4], this leads, for $i = 1, \dots, k$, to

$$\pi_\tau(\Phi_i(\eta,\theta)) \underbrace{\boldsymbol{\epsilon}(\Phi_i(\eta,\theta))^{-1}\boldsymbol{D}(t,\Phi_i(\eta,\theta))}_{=\boldsymbol{E}(t,\Phi_i(\eta,\theta))} = \nabla\Phi_i(\eta,\theta)^\dagger \begin{pmatrix} \boldsymbol{E}_{i,\mathrm{ext}}(t,\eta) \\ 0 \end{pmatrix}. \quad (13)$$

3. Analysis of the coupling conditions

The coupling relations (12) and (13) between the tangential traces of the electromagnetic fields and the external quantities $\boldsymbol{I}_{\mathrm{ext}}$ and $\boldsymbol{E}_{\mathrm{ext}}$ can be expressed by the operators P_{mag} and P_{el} via

$$\boldsymbol{I}_{\mathrm{ext}}(t,\cdot) = P_{\mathrm{mag}}\big(\nu(\cdot) \times \boldsymbol{\mu}(\cdot)^{-1}\boldsymbol{B}(t,\cdot)\big), \quad \boldsymbol{E}_{\mathrm{ext}}(t,\cdot) = P_{\mathrm{el}}\big(\pi_\tau(\cdot)\boldsymbol{\epsilon}(\cdot)^{-1}\boldsymbol{D}(t,\cdot)\big). \quad (14)$$

In this section, we provide the functional-analytic framework required for a rigorous treatment of these coupling conditions. We first introduce the boundary trace spaces for the electric and magnetic fields on the cable surfaces and then analyze the operators P_{mag} and P_{el}, clarifying their mutual relation.

3.1. **Tangential boundary trace spaces.** As mentioned in Section 2.1, the spatial domain where the electromagnetic field evolves is represented as $\Omega = \Omega_0 \setminus \bigcup_{i=1}^k \overline{\Omega_i}$, and the boundary is divided into $\partial\Omega = \Gamma_{\mathrm{lat}} \cup \Gamma_{\mathrm{r}}$, where $\Gamma_{\mathrm{r}} = \Gamma_{\mathrm{end}} \cup \Gamma_{\mathrm{ext}}$ with

$$\Gamma_{\mathrm{lat}} = \bigcup_{i=1}^k \Gamma_{i,\mathrm{lat}}, \quad \Gamma_{\mathrm{end}} = \bigcup_{i=1}^k \Gamma_{i,\mathrm{end}}, \quad \Gamma_{\mathrm{ext}} = \partial\Omega_0,$$

i.e., Γ_{lat} is the union of the lateral surfaces of the cables, Γ_{end} is the union of the end faces of the cables, Γ_{ext} is the exterior boundary of the computational domain and Γ_{r} is the rest of boundary from the perspective of Γ_{lat}, see Figure 1a. Motivated by the zero tangential boundary conditions for the electric field on Γ_{r}, we introduce the space of smooth functions with compact support not intersecting with Γ_r, i.e.,

$$\mathring{\mathrm{C}}^\infty_{\Gamma_{\mathrm{r}}}(\Omega) := \big\{ f \in \mathrm{C}^\infty(\overline{\Omega}) \,\big|\, \operatorname{supp} f \subseteq \overline{\Omega} \text{ compact with } \overline{\Gamma_{\mathrm{r}}} \cap \operatorname{supp} f = \emptyset \big\}.$$

Hence, $f \in \mathring{\mathrm{C}}^\infty_{\Gamma_{\mathrm{r}}}(\Omega)$ is always zero on Γ_{r}, but can be non-zero on $\partial\Omega \setminus \Gamma_{\mathrm{r}}$, and let $\mathring{\mathrm{C}}^\infty(\Omega) := \mathring{\mathrm{C}}^\infty_{\partial\Omega}(\Omega)$. As for the other function spaces in this work, we append "$;\mathbb{C}^m$"

to denote the space of $\mathbb{C}^m$-valued functions being component-wise within these spaces. Also recall the Sobolev space for the distributional rot-operator, namely

$$\mathrm{H}(\mathrm{rot},\Omega) = \{\boldsymbol{E} \in \mathrm{L}^2(\Omega;\mathbb{C}^3) \,|\, \mathrm{rot}\,\boldsymbol{E} \in \mathrm{L}^2(\Omega;\mathbb{C}^3)\},$$

which is endowed with the inner product

$$\langle \boldsymbol{E},\boldsymbol{H}\rangle_{\mathrm{H}(\mathrm{rot},\Omega)} \coloneqq \langle \boldsymbol{E},\boldsymbol{H}\rangle_{\mathrm{L}^2(\Omega;\mathbb{C}^3)} + \langle \mathrm{rot}\,\boldsymbol{E},\mathrm{rot}\,\boldsymbol{H}\rangle_{\mathrm{L}^2(\Omega;\mathbb{C}^3)}.$$

We define a subspace of $\mathrm{H}(\mathrm{rot},\Omega)$ that has homogeneous boundary conditions on Γ_r by the closure of $\mathring{\mathrm{C}}^\infty_{\Gamma_\mathrm{r}}(\Omega)$ with respect to the norm in $\mathrm{H}(\mathrm{rot},\Omega)$, i.e.,

$$\mathring{\mathrm{H}}_{\Gamma_\mathrm{r}}(\mathrm{rot},\Omega) \coloneqq \overline{\mathring{\mathrm{C}}^\infty_{\Gamma_\mathrm{r}}(\Omega;\mathbb{C}^3)}^{\mathrm{H}(\mathrm{rot},\Omega)}$$

Motivated by the boundary conditions discussed in Section 2.3 for Maxwell's equations, we consider two specific types of boundary trace operators, namely

$$\pi_\tau : \begin{cases} \mathrm{C}^\infty(\overline{\Omega};\mathbb{C}^3) & \to \quad \mathrm{L}^2(\partial\Omega;\mathbb{C}^3), \\ \qquad\quad \boldsymbol{E} & \mapsto \quad (\nu \times (\boldsymbol{E}|_{\partial\Omega})) \times \nu, \end{cases} \qquad \gamma_\tau : \begin{cases} \mathrm{C}^\infty(\overline{\Omega};\mathbb{C}^3) & \to \quad \mathrm{L}^2(\partial\Omega;\mathbb{C}^3), \\ \qquad\quad \boldsymbol{H} & \mapsto \quad \nu \times (\boldsymbol{H}|_{\partial\Omega}), \end{cases}$$

where $\nu \in \mathrm{L}^\infty(\partial\Omega;\mathbb{C}^3)$ pointwisely refers to the outward normal vector on $\partial\Omega$. To properly define the boundary spaces, one would need to introduce several technical concepts that could make the presentation less accessible. To keep the exposition concise, we therefore adopt a simplified approach. The downside is that certain structural aspects of the coupling are not immediately apparent in its operator-theoretic representation, since its domain cannot be characterized explicitly without introducing these additional spaces. The integration by parts formula for the rot operator [GR86, Thm. 2.11] gives for all $\boldsymbol{E} \in \mathring{\mathrm{C}}^\infty_{\Gamma_\mathrm{r}}(\Omega;\mathbb{C}^3)$, $\boldsymbol{H} \in \mathrm{C}^\infty(\overline{\Omega};\mathbb{C}^3)$,

$$\langle \mathrm{rot}\,\boldsymbol{E},\boldsymbol{H}\rangle_{\mathrm{L}^2(\Omega;\mathbb{C}^3)} - \langle \boldsymbol{E},\mathrm{rot}\,\boldsymbol{H}\rangle_{\mathrm{L}^2(\Omega;\mathbb{C}^3)} = \langle \gamma_\tau\boldsymbol{E},\pi_\tau\boldsymbol{H}\rangle_{\mathrm{L}^2(\Gamma_{\mathrm{lat}};\mathbb{C}^3)} \tag{15}$$

Now it is possible to extend this integration by parts formula for $\boldsymbol{E},\boldsymbol{H} \in \mathrm{H}(\mathrm{rot},\Omega)$ with suitable boundary spaces. However, the left-hand-side of (15) is well defined for $\boldsymbol{E} \in \mathring{\mathrm{H}}_{\Gamma_\mathrm{r}}(\mathrm{rot},\Omega)$ and $\boldsymbol{H} \in \mathrm{H}(\mathrm{rot},\Omega)$. Hence, we extend the right-hand-side $\langle \gamma_\tau\boldsymbol{E},\pi_\tau\boldsymbol{H}\rangle_{\mathrm{L}^2(\Gamma_{\mathrm{lat}};\mathbb{C}^3)}$ just by the left-hand-side of (15) and denote it as

$$\langle \gamma_\tau\boldsymbol{E},\pi_\tau\boldsymbol{H}\rangle_{\mathcal{V}_\tau^\times,\mathcal{V}_\tau} \coloneqq \langle \mathrm{rot}\,\boldsymbol{E},\boldsymbol{H}\rangle_{\mathrm{L}^2(\Omega;\mathbb{C}^3)} - \langle \boldsymbol{E},\mathrm{rot}\,\boldsymbol{H}\rangle_{\mathrm{L}^2(\Omega;\mathbb{C}^3)}.$$

We will regard the left-hand-side of the previous equation just as symbol, although it possible to define spaces $\mathcal{V}_\tau$ and $\mathcal{V}_\tau^\times$, and extensions of π_τ and γ_τ to $\mathrm{H}(\mathrm{rot},\Omega)$ such that this is really a dual pairing, see e.g., [BCS02] or [Skr21b] for a more general approach. We further define $\langle \pi_\tau\boldsymbol{H},\gamma_\tau\boldsymbol{E}\rangle_{\mathcal{V}_\tau,\mathcal{V}_\tau^\times} = \overline{\langle \gamma_\tau\boldsymbol{E},\pi_\tau\boldsymbol{H}\rangle_{\mathcal{V}_\tau^\times,\mathcal{V}_\tau}}$, which gives

$$\langle \pi_\tau\boldsymbol{H},\gamma_\tau\boldsymbol{E}\rangle_{\mathcal{V}_\tau,\mathcal{V}_\tau^\times} = \langle \boldsymbol{H},\mathrm{rot}\,\boldsymbol{E}\rangle_{\mathrm{L}^2(\Omega;\mathbb{C}^3)} - \langle \mathrm{rot}\,\boldsymbol{H},\boldsymbol{E}\rangle_{\mathrm{L}^2(\Omega;\mathbb{C}^3)}.$$

Definition 3.1. Suppose that the spatial domains are as in Section 2.1. Let $f \in \mathrm{L}^2(\Gamma_{\mathrm{lat}};\mathbb{C}^3)$ and $\boldsymbol{E} \in \mathring{\mathrm{H}}_{\Gamma_\mathrm{r}}(\mathrm{rot},\Omega)$. We say $\pi_\tau\boldsymbol{E} = f$ on Γ_{lat}, if

$$\langle \pi_\tau\boldsymbol{E},\gamma_\tau\boldsymbol{H}\rangle_{\mathcal{V}_\tau,\mathcal{V}_\tau^\times} = \langle f,\gamma_\tau\boldsymbol{H}\rangle_{\mathrm{L}^2(\Gamma_{\mathrm{lat}})} \quad \text{for all} \quad \boldsymbol{H} \in \mathrm{C}^\infty(\overline{\Omega};\mathbb{C}^3),$$

i.e., f is the weak L^2 representative of $\pi_\tau\boldsymbol{E}$.

A concise overview of these boundary spaces is provided in the appendix of [SW24]. Since their detailed structure is not required in the present work, it is not discussed here in detail.

3.2. **The port operators.** Next we provide a rigorous functional-analytic treatment of the port operators P_{mag} and P_{el}, which were previously introduced in Section 2.4 at a formal level. Namely, for the parameterization Φ_i of the lateral boundary $\Gamma_{i,\mathrm{lat}}$ of the ith cable (see (1)), we denote the inverse by Ψ_i. That is,

$$\Psi_i\colon \begin{cases} \Gamma_{i,\mathrm{lat}} & \to & [0,1]\times(-\pi,\pi], \\ \Phi(\eta,\theta) & \mapsto & (\eta,\theta), \end{cases} \qquad \Psi_{i1}\colon \begin{cases} \Gamma_{i,\mathrm{lat}} & \to & [0,1], \\ \Phi(\eta,\theta) & \mapsto & \eta. \end{cases}$$

We introduce the operators

$$P_{i,\mathrm{mag}}\colon \begin{cases} \mathrm{L}^2(\Gamma_{i,\mathrm{lat}};\mathbb{C}^3) & \to & \mathrm{L}^2((0,1);\mathbb{C}), \\ g & \mapsto & \left(\eta\mapsto \oint\limits_{\alpha_i(\eta)+\beta_{i\eta}} g\times\nu\cdot \mathrm{d}s\right), \end{cases}$$

$$P_{i,\mathrm{el}}\colon \begin{cases} \mathrm{L}^2((0,1);\mathbb{C}) & \to & \mathrm{L}^2(\Gamma_{i,\mathrm{lat}};\mathbb{C}^3), \\ f & \mapsto & (\nabla\Phi_i\circ\Psi_i)^{\dagger}\begin{pmatrix} f\circ\Psi_{i1} \\ 0\end{pmatrix}. \end{cases}$$

The coupling conditions (12) and (13) can now be reformulated to (14) via

$$P_{\mathrm{mag}}\colon \begin{cases} \mathrm{L}^2(\Gamma_{\mathrm{lat}};\mathbb{C}^3) & \to & \mathrm{L}^2((0,1);\mathbb{C}^k), \\ g & \mapsto & \begin{pmatrix} P_{1,\mathrm{mag}}\left(g|_{\Gamma_{1,\mathrm{lat}}}\right) \\ \vdots \\ P_{k,\mathrm{mag}}\left(g|_{\Gamma_{k,\mathrm{lat}}}\right)\end{pmatrix}, \end{cases} \tag{16}$$

$$P_{\mathrm{el}}\colon \begin{cases} \mathrm{L}^2((0,1);\mathbb{C}^k) & \to & \mathrm{L}^2(\Gamma_{\mathrm{lat}};\mathbb{C}^3), \\ \begin{pmatrix} f_1\\ \vdots \\ f_k\end{pmatrix} & \mapsto & g \quad\text{with}\quad g|_{\Gamma_{i,\mathrm{lat}}} = P_{i,\mathrm{el}} f_i \quad \forall\, i=1,\dots,k. \end{cases} \tag{17}$$

A key property for our subsequent analysis is that these two operators are adjoint to each other, a fact that was established in [CGRS25] in the context of deriving an energy balance.

Proposition 3.2 ([CGRS25, Prop. 4.2]). *The operators in* (16) *and* (17) *fulfill*

$$P_{\mathrm{el}}^* = P_{\mathrm{mag}}.$$

A schematic representation of the overall coupling structure is given in Figure 2.

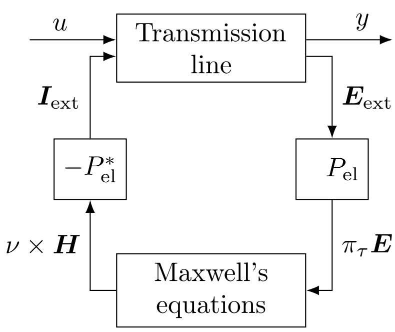


FIGURE 2. Coupled ports

3.3. **Lifting of a $\mathrm{L}^2(0,1)$ function to $\mathrm{H}(\mathrm{rot},\Omega)$.** Note that the coupling operator P_{el} maps into $\mathrm{L}^2(\Gamma_{\mathrm{lat}})$, and we aim to impose the coupling condition $P_{\mathrm{el}}\boldsymbol{E}_{\mathrm{ext}} = \pi_\tau \boldsymbol{E}$. However, the tangential trace operator π_τ maps to a space which is neither a subset nor a superset of $\mathrm{L}^2(\Gamma_{\mathrm{lat}})$, see [BCS02]. Hence, one must ensure that the coupling condition is well-defined, that is, for a given output $\boldsymbol{E}_{\mathrm{ext}}$ of the transmission line, there exists a field $\boldsymbol{E} \in \mathrm{H}(\mathrm{rot},\Omega)$ satisfying $P_{\mathrm{el}}\boldsymbol{E}_{\mathrm{ext}} = \pi_\tau \boldsymbol{E}$.

Note that the output $\boldsymbol{E}_{\mathrm{ext}} = \partial_\eta(\boldsymbol{C}^{-1}\boldsymbol{q}) = \partial_\eta \boldsymbol{V}$ employs the derivative of a part of the state of the transmission lines. Furthermore, $P_{\mathrm{el}}\boldsymbol{E}_{\mathrm{ext}}$ can be viewed as the result of applying the chain rule, representing the tangential derivative (or gradient) of $\boldsymbol{V} \circ \Psi$. This observation suggests that $P_{\mathrm{el}}\boldsymbol{E}_{\mathrm{ext}}$ provides a natural candidate for extension from the boundary Γ_{lat} into the domain Ω.

Note that we can extend the paths α_i in both direction by an $\epsilon > 0$ to a path $\hat{\alpha}_i \in \mathrm{C}^2((-\epsilon, 1+\epsilon))$. This allows us to extend Φ_i to a C^2 diffeomorphism (onto its image) by

$$\hat{\Phi}_i \colon \left\{ \begin{array}{rcl} (-\epsilon, 1+\epsilon) \times (-\pi,\pi) \times (-\epsilon,\epsilon) & \to & \mathbb{R}^3, \\ (\eta,\theta,s) & \mapsto & \hat{\alpha}_i(\eta) + (1+s)\beta_{i\eta}(\theta), \end{array} \right. \tag{18}$$

for $\epsilon > 0$ sufficiently small. We will denote the inverse of $\hat{\Phi}_i$ by

$$\hat{\Psi}_i \colon \operatorname{ran}\hat{\Phi}_i \to (-\epsilon, 1+\epsilon) \times (-\pi,\pi) \times (-\epsilon,\epsilon).$$

Clearly $\hat{\Psi}_i = \Psi_i$ on $\Gamma_{i,\mathrm{lat}}$. Hence,[1] (by the chain rule $(\nabla\hat{\Phi}_i)^{-1} \circ \hat{\Psi}_i = \nabla\hat{\Psi}_i$)

$$\begin{aligned} P_{i,\mathrm{el}}\boldsymbol{E}_{\mathrm{ext}} = P_{i,\mathrm{el}}\partial_\eta \boldsymbol{V} &= (\nabla\Phi_i)^\dagger \begin{pmatrix} \partial_\eta \boldsymbol{V} \\ 0 \end{pmatrix} \circ \Psi_i \\ &= (\nabla\hat{\Phi}_i)^{-1} \begin{pmatrix} \partial_\eta \boldsymbol{V} \\ 0 \\ 0 \end{pmatrix} \circ \hat{\Psi}_i\big|_{\Gamma_{i,\mathrm{lat}}} = \nabla(\boldsymbol{V} \circ \hat{\Psi}_i)\big|_{\Gamma_{i,\mathrm{lat}}}. \end{aligned} \tag{19}$$

Note that $\nabla(\boldsymbol{V} \circ \hat{\Psi}_i)$ has vanishing rot as a gradient field. So in order to extend it from $\operatorname{ran}\hat{\Phi}_i$ we multiply it by a cut-off function $\chi_i \in \mathring{\mathrm{C}}^\infty(\operatorname{ran}\hat{\Phi}_i)$ that is 1 on $\Gamma_{i,\mathrm{lat}}$. Then we can extend $\chi_i\nabla(\boldsymbol{V} \circ \hat{\Psi}_i)$ by zero outside of $\operatorname{ran}\hat{\Phi}_i$ and obtain $\chi_i\nabla(\boldsymbol{V} \circ \hat{\Psi}_i) \in \mathrm{H}(\mathrm{rot},\Omega)$ by the product rule for rot.

Lemma 3.3. *The mapping*

$$\hat{P}_{i,\mathrm{el}} \colon \left\{ \begin{array}{rcl} \mathrm{H}^1((0,1)) & \to & \mathrm{H}(\mathrm{rot},\Omega), \\ \boldsymbol{V} & \mapsto & \chi_i \nabla\hat{\Psi}_i \begin{pmatrix} (\partial_\eta \boldsymbol{V}) \circ \hat{\Psi}_i \\ 0 \\ 0 \end{pmatrix} \end{array} \right.$$

is bounded and satisfies $\hat{P}_{i,\mathrm{el}}\boldsymbol{V}\big|_{\Gamma_{i,\mathrm{lat}}} = P_{i,\mathrm{el}}\big(\partial_\eta \boldsymbol{V}\big)$. *In particular,* $\hat{P}_{i,\mathrm{el}}\boldsymbol{V} \in \mathrm{H}(\mathrm{rot},\Omega)$ *such that* $\pi_\tau \hat{P}_{i,\mathrm{el}}\boldsymbol{V} = P_{i,\mathrm{el}}\big(\partial_\eta \boldsymbol{V}\big)$.

Proof. The equality $\hat{P}_{i,\mathrm{el}}\boldsymbol{V}\big|_{\Gamma_{i,\mathrm{lat}}} = P_{i,\mathrm{el}}\partial_\eta \boldsymbol{V}$ follows from (19) and the boundedness from

$$\begin{aligned} \|\chi_i\nabla(\boldsymbol{V} \circ \hat{\Psi}_i)\|^2_{\mathrm{H}(\mathrm{rot},\Omega)} &= \|\chi_i\nabla(\boldsymbol{V} \circ \hat{\Psi}_i)\|^2_{\mathrm{L}^2(\Omega)} + \|\nabla\chi_i \times \nabla(\boldsymbol{V} \circ \hat{\Psi}_i)\|^2_{\mathrm{L}^2(\Omega)} \\ &\leq (\|\chi_i\|^2_\infty + \|\nabla\chi_i\|^2_\infty)\|\nabla(\boldsymbol{V} \circ \hat{\Psi}_i)\|^2_{\mathrm{L}^2(\operatorname{ran}\hat{\Phi}_i)} \\ &\leq C_i(\|\chi_i\|^2_\infty + \|\nabla\chi_i\|^2_\infty)\|\nabla\hat{\Psi}_i\|^2_\infty \|\partial_\eta \boldsymbol{V}\|^2_{\mathrm{L}^2((0,1))} \\ &\leq C_i(\|\chi_i\|^2_\infty + \|\nabla\chi_i\|^2_\infty)\|\nabla\hat{\Psi}_i\|^2_\infty \|\boldsymbol{V}\|^2_{\mathrm{H}^1((0,1))}, \end{aligned}$$

where C_i is a suitable constant that corresponds to the cylindrical coordinates imposed by $\hat{\Phi}_i$ ❑

[1] In order to simplify notation we say $\boldsymbol{V}(\eta,\theta,s) \coloneqq \boldsymbol{V}(\eta)$ to make compositions work.

4. Analysis of the coupled system

4.1. **Formulation as an infinite-dimensional system.** For our coupled system we regard the *state* $x(t)$ and *effort* $e(t)$, which are given by

$$e(t) = \begin{pmatrix} \boldsymbol{I}(t,\cdot) \\ \boldsymbol{H}(t,\cdot) \\ \boldsymbol{V}(t,\cdot) \\ \boldsymbol{E}(t,\cdot) \end{pmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{L}(\cdot)^{-1} & 0 & 0 & 0 \\ 0 & \boldsymbol{\mu}(\cdot)^{-1} & 0 & 0 \\ 0 & 0 & \boldsymbol{C}(\cdot)^{-1} & 0 \\ 0 & 0 & 0 & \boldsymbol{\epsilon}(\cdot)^{-1} \end{bmatrix}}_{=\mathcal{H}} \underbrace{\begin{pmatrix} \boldsymbol{\psi}(t,\cdot) \\ \boldsymbol{B}(t,\cdot) \\ \boldsymbol{q}(t,\cdot) \\ \boldsymbol{D}(t,\cdot) \end{pmatrix}}_{=x(t)}, \tag{20a}$$

and evolve in the *state space*

$$\mathcal{X} = \begin{bmatrix} \mathrm{L}^2((0,1);\mathbb{C}^k) \\ \mathrm{L}^2(\Omega;\mathbb{C}^3) \\ \mathrm{L}^2((0,1);\mathbb{C}^k) \\ \mathrm{L}^2(\Omega;\mathbb{C}^3) \end{bmatrix}_\times \tag{20b}$$

endowed with the canonical inner product. Then, by using Assumption 2.1 and Assumption 2.4, $\mathcal{H}$ defines a bounded and bijective operator on $\mathcal{X}$ via pointwise multiplication, and a bounded *damping operator*

$$\mathfrak{R} = \begin{bmatrix} \boldsymbol{R}(\cdot) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{G}(\cdot) & 0 \\ 0 & 0 & 0 & \boldsymbol{\sigma}(\cdot) \end{bmatrix}. \tag{20c}$$

In order to give a vague understanding of our goals we are a little bit imprecise in the following few lines. Our coupled system is defined by the differential operator (specifically, its operator closure)

$$\mathfrak{J} = \begin{bmatrix} 0 & 0 & -\frac{\mathrm{d}}{\mathrm{d}\eta} & 0 \\ 0 & 0 & 0 & -\operatorname{rot} \\ -\frac{\mathrm{d}}{\mathrm{d}\eta} & \frac{\mathrm{d}}{\mathrm{d}\eta} P_{\mathrm{el}}^{*}\gamma_\tau & 0 & 0 \\ 0 & \operatorname{rot} & 0 & 0 \end{bmatrix},$$

$$\operatorname{dom}\mathfrak{J} = \left\{ \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{H} \\ \boldsymbol{V} \\ \boldsymbol{E} \end{pmatrix} \in \begin{bmatrix} \mathrm{H}^1((0,1);\mathbb{C}^k) \\ \mathrm{C}^\infty(\overline{\Omega};\mathbb{C}^3) \\ \mathrm{H}^1((0,1);\mathbb{C}^k) \\ \mathring{\mathrm{H}}_{\Gamma_\mathrm{r}}(\operatorname{rot},\Omega) \end{bmatrix}_\times \,\middle|\, \pi_\tau \boldsymbol{E} = P_{\mathrm{el}} \tfrac{\mathrm{d}}{\mathrm{d}\eta}\boldsymbol{V} \text{ on } \Gamma_{\mathrm{lat}} \right\}$$

through the (misleadingly appearing as autonomous) differential equation

$$\dot{x} = (\mathfrak{J} - \mathfrak{R})\mathcal{H}x.$$

Note that the boundary condition $\pi_\tau \boldsymbol{E}\big|_{\Gamma_\mathrm{r}} = 0$ (which comprises (10) and (11)) is encoded in $\boldsymbol{E} \in \mathring{\mathrm{C}}^\infty_{\Gamma_\mathrm{r}}(\Omega;\mathbb{C}^3)$. The operator closure does not alter this tangential boundary condition, since the tangential trace is continuous w.r.t. $\|\cdot\|_{\mathrm{H}(\operatorname{rot},\Omega)}$.

Let two efforts $e^i = \big(\boldsymbol{I}^i, \boldsymbol{H}^i, \boldsymbol{V}^i, \boldsymbol{E}^i\big)$, $i = 1,2$, be given (we use row-vector notation for layout reasons). Using (3), the corresponding total currents are defined as $\boldsymbol{I}^i_{\mathrm{tot}} \coloneqq \boldsymbol{I}^i - P_{\mathrm{el}}^{*}\boldsymbol{H}^i$. Our next objective is to show that a boundary triple (see Definition A.2) can be associated with the operator $\mathfrak{J}$. In particular, for $e^1, e^2 \in \operatorname{dom}\mathfrak{J}$, partitioned and denoted as in (20a), we aim to establish the abstract

Green identity

$$
\begin{aligned}
&\langle \mathfrak{J}e^1, e^2\rangle_{\mathcal{X}} + \langle e^1, \mathfrak{J}e^2\rangle_{\mathcal{X}} \\
&\qquad = -\langle \boldsymbol{V}^1(1), \boldsymbol{I}^2_{\mathrm{tot}}(1)\rangle_{\mathbb{C}^k} + \langle \boldsymbol{V}^1(0), \boldsymbol{I}^2_{\mathrm{tot}}(0)\rangle_{\mathbb{C}^k} \\
&\qquad\qquad - \langle \boldsymbol{I}^1_{\mathrm{tot}}(1), \boldsymbol{V}^2(1)\rangle_{\mathbb{C}^k} + \langle \boldsymbol{I}^1_{\mathrm{tot}}(0), \boldsymbol{V}^2(0)\rangle_{\mathbb{C}^k} \\
&\qquad = \left\langle \begin{pmatrix} \boldsymbol{V}^1(0) \\ -\boldsymbol{V}^1(1) \end{pmatrix}, \begin{pmatrix} \boldsymbol{I}^2_{\mathrm{tot}}(0) \\ \boldsymbol{I}^2_{\mathrm{tot}}(1) \end{pmatrix} \right\rangle_{\mathbb{C}^{2k}} + \left\langle \begin{pmatrix} \boldsymbol{I}^1_{\mathrm{tot}}(0) \\ \boldsymbol{I}^1_{\mathrm{tot}}(1) \end{pmatrix}, \begin{pmatrix} \boldsymbol{V}^2(0) \\ -\boldsymbol{V}^2(1) \end{pmatrix} \right\rangle_{\mathbb{C}^{2k}}.
\end{aligned}
$$

4.2. **Boundary triples and the semigroup property.** In order to properly define boundary conditions that admit existence and uniqueness of solutions we analyze the blocks of the operator $\mathfrak{J}$. We define

$$
\mathfrak{A}_1 = \begin{bmatrix} -\frac{\mathrm{d}}{\mathrm{d}\eta} & \frac{\mathrm{d}}{\mathrm{d}\eta} P_{\mathrm{el}}^* \gamma_\tau \\ 0 & \operatorname{rot} \end{bmatrix}, \ \mathfrak{A}_2 = \begin{bmatrix} -\frac{\mathrm{d}}{\mathrm{d}\eta} & 0 \\ 0 & -\operatorname{rot} \end{bmatrix}, \ \mathcal{X}_1 = \begin{bmatrix} \mathrm{L}^2((0,1);\mathbb{C}^k) \\ \mathrm{L}^2(\Omega;\mathbb{C}^3) \end{bmatrix}_{\times},
$$

$$
\begin{aligned}
\operatorname{dom} \mathfrak{A}_1 &= \left\{ \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{H} \end{pmatrix} \in \begin{bmatrix} \mathrm{H}^1((0,1);\mathbb{C}^k) \\ \mathrm{C}^\infty(\overline{\Omega};\mathbb{C}^3) \end{bmatrix}_{\times} \right\} \subseteq \mathcal{X}_1 \\
\operatorname{dom} \mathfrak{A}_2 &= \left\{ \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{E} \end{pmatrix} \in \begin{bmatrix} \mathrm{H}^1((0,1);\mathbb{C}^k) \\ \mathring{\mathrm{H}}_{\Gamma_{\mathrm{r}}}(\operatorname{rot},\Omega) \end{bmatrix}_{\times} \,\middle|\, \pi_\tau \boldsymbol{E} = P_{\mathrm{el}} \tfrac{\mathrm{d}}{\mathrm{d}\eta} \boldsymbol{V} \text{ on } \Gamma_{\mathrm{lat}} \right\} \subseteq \mathcal{X}_1,
\end{aligned} \tag{21a}
$$

By invoking that $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_1$, the differential operator $\mathfrak{J}$ can now be written as

$$
\mathfrak{J} = \begin{bmatrix} 0 & \mathfrak{A}_2 \\ \mathfrak{A}_1 & 0 \end{bmatrix}. \tag{21b}
$$

We obviously have that $\operatorname{dom} \mathfrak{J} = \operatorname{dom} \mathfrak{A}_1 \times \operatorname{dom} \mathfrak{A}_2$. Moreover, we introduce the following operators that will form our minimal operator

$$
\mathring{\mathfrak{A}}_1 = \begin{bmatrix} -\frac{\mathrm{d}}{\mathrm{d}\eta} & \frac{\mathrm{d}}{\mathrm{d}\eta} P_{\mathrm{el}}^* \gamma_\tau \\ 0 & \operatorname{rot} \end{bmatrix} \quad \text{and} \quad \mathring{\mathfrak{A}}_2 = \begin{bmatrix} -\frac{\mathrm{d}}{\mathrm{d}\eta} & 0 \\ 0 & -\operatorname{rot} \end{bmatrix} \tag{22a}
$$

with domains

$$
\begin{aligned}
\operatorname{dom} \mathring{\mathfrak{A}}_1 &= \left\{ \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{H} \end{pmatrix} \in \begin{bmatrix} \mathrm{H}^1((0,1);\mathbb{C}^k) \\ \mathrm{C}^\infty(\overline{\Omega};\mathbb{C}^3) \end{bmatrix}_{\times} \,\middle|\, \boldsymbol{I} - P_{\mathrm{el}}^* \gamma_\tau \boldsymbol{H} \in \mathring{\mathrm{H}}^1((0,1);\mathbb{C}^k) \right\}, \\
\operatorname{dom} \mathring{\mathfrak{A}}_2 &= \left\{ \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{E} \end{pmatrix} \in \begin{bmatrix} \mathring{\mathrm{H}}^1((0,1);\mathbb{C}^k) \\ \mathring{\mathrm{H}}_{\Gamma_{\mathrm{r}}}(\operatorname{rot},\Omega) \end{bmatrix}_{\times} \,\middle|\, \pi_\tau \boldsymbol{E} = P_{\mathrm{el}} \tfrac{\mathrm{d}}{\mathrm{d}\eta} \boldsymbol{V} \right\}.
\end{aligned} \tag{22b}
$$

Note that by definition we have

$$
-\mathring{\mathfrak{A}}_1 \subseteq \mathfrak{A}_1 \quad \text{and} \quad -\mathring{\mathfrak{A}}_2 \subseteq \mathfrak{A}_2. \tag{23}
$$

Lemma 4.1. *Suppose that the spatial domains are as in Section 2.1. Then the operators $\mathring{\mathfrak{A}}_1$, $\mathring{\mathfrak{A}}_2$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ as in (21a) and (22) are densely defined.*

Proof. It is enough to show that $\mathring{\mathfrak{A}}_1$ and $\mathring{\mathfrak{A}}_2$ are densely defined as their domains are subsets of $\operatorname{dom}(\mathfrak{A}_1)$ and $\operatorname{dom}(\mathfrak{A}_2)$, respectively. Moreover, $\operatorname{dom} \mathring{\mathfrak{A}}_1$ contains $\mathring{\mathrm{C}}^\infty((0,1);\mathbb{C}^k) \times \mathring{\mathrm{C}}^\infty(\Omega;\mathbb{C}^3)$ and is therefore dense in $\mathrm{L}^2((0,1);\mathbb{C}^k) \times \mathrm{L}^2(\Omega;\mathbb{C}^3)$. Hence, it is left to show that $\operatorname{dom} \mathring{\mathfrak{A}}_2$ is dense in $\mathrm{L}^2((0,1);\mathbb{C}^k) \times \mathrm{L}^2(\Omega;\mathbb{C}^3)$.

Let $\left(\begin{smallmatrix} \boldsymbol{V} \\ \boldsymbol{E} \end{smallmatrix}\right) \in \mathrm{L}^2((0,1);\mathbb{C}^k) \times \mathrm{L}^2(\Omega;\mathbb{C}^3)$. Then there exists a sequence $(\boldsymbol{V}_n)_{n\in\mathbb{N}}$ in $\mathring{\mathrm{C}}^\infty((0,1))$ that converges to $\boldsymbol{V}$ w.r.t. $\|\cdot\|_{\mathrm{L}^2((0,1))}$. We use the mapping $\hat{P}_{\mathrm{el}}$ from Lemma 3.3 to lift $\boldsymbol{V}_n$ on $\mathrm{H}(\operatorname{rot},\Omega)$. Multiplying this by a sequence $(\chi_n)_{n\in\mathbb{N}}$ of $\mathring{\mathrm{C}}^\infty$ cut-off functions that are 1 on Γ_{lat} and satisfy $\|\chi_n\|_{\mathrm{L}^2} \leq \frac{1}{n} \frac{1}{\|\partial_\eta \boldsymbol{V}_n\|_\infty}$, we define

$$
\boldsymbol{E}_n^1 \coloneqq \chi_n \hat{P}_{\mathrm{el}} \boldsymbol{V}_n,
$$

which satisfies the boundary condition $\pi_\tau \boldsymbol{E}_n^1 = P_{\mathrm{el}} \boldsymbol{V}_n$ and

$$
\|\boldsymbol{E}_n^1\|_{\mathrm{L}^2(\Omega)} \leq \|\chi_n\|_{\mathrm{L}^2(\Omega)} \|\partial_\eta \boldsymbol{V}_n\|_\infty \to 0.
$$

For the given $\boldsymbol{E}$ there exists a sequence $(\boldsymbol{E}_n^2)_{n\in\mathbb{N}}$ in $\mathring{\mathrm{C}}^\infty(\Omega)$ that converges to $\boldsymbol{E}$ w.r.t. $\|\cdot\|_{\mathrm{L}^2(\Omega)}$. Hence, we define

$$\boldsymbol{E}_n := \boldsymbol{E}_n^1 + \boldsymbol{E}_n^2.$$

This makes sure that $\left(\begin{smallmatrix}\boldsymbol{V}_n\\ \boldsymbol{E}_n\end{smallmatrix}\right) \in \operatorname{dom}\mathring{\mathfrak{A}}_1$ and $\left(\begin{smallmatrix}\boldsymbol{V}_n\\ \boldsymbol{E}_n\end{smallmatrix}\right) \to \left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right)$. ❑

Lemma 4.2. *Suppose that the spatial domains are as in Section 2.1. Then the operators as in (21a) and (22) fulfill*

$$\mathring{\mathfrak{A}}_1^* = -\mathfrak{A}_2 \quad \text{and} \quad \mathfrak{A}_1^* = -\mathring{\mathfrak{A}}_2.$$

Proof. We first show $\mathring{\mathfrak{A}}_1^* = -\mathfrak{A}_2$. For $\left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right) \in \operatorname{dom}\mathring{\mathfrak{A}}_1^*$ we have (by definition of the adjoint)

$$\left\langle \mathring{\mathfrak{A}}_1^*\left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right), \left(\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{smallmatrix}\right)\right\rangle_{\mathcal{X}_1} = \left\langle \left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right), \mathring{\mathfrak{A}}_1\left(\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{smallmatrix}\right)\right\rangle_{\mathcal{X}_1} \quad \text{for all} \quad \left(\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{smallmatrix}\right) \in \operatorname{dom}\mathring{\mathfrak{A}}_1. \tag{24}$$

Hence, we can calculate the action of $\mathring{\mathfrak{A}}_1^*$ by testing component-wise.

- For every $\boldsymbol{I} \in \mathring{\mathrm{C}}^\infty((0,1);\mathbb{C}^k)$ we have $\left(\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{0}\end{smallmatrix}\right) \in \operatorname{dom}\mathring{\mathfrak{A}}_1$ and
$$\left\langle \pi_1\mathring{\mathfrak{A}}_1^*\left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right), \boldsymbol{I}\right\rangle_{\mathcal{X}_1} = \left\langle \mathring{\mathfrak{A}}_1^*\left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right), \left(\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{0}\end{smallmatrix}\right)\right\rangle_{\mathcal{X}_1} \overset{(24)}{=} \left\langle \boldsymbol{V}, -\tfrac{\mathrm{d}}{\mathrm{d}\eta}\boldsymbol{I}\right\rangle_{\mathrm{L}^2((0,1);\mathbb{C}^k)}, \tag{25}$$
where π_1 is the projection on the first component (the $\mathrm{L}^2((0,1);\mathbb{C}^k)$ component). Hence, $\boldsymbol{I} \in \mathrm{H}^1((0,1);\mathbb{C}^k)$ and $\pi_1\mathring{\mathfrak{A}}_1^*\left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right) = \frac{\mathrm{d}}{\mathrm{d}\eta}\boldsymbol{V}$.
- For every $\boldsymbol{H} \in \mathring{\mathrm{C}}^\infty(\Omega;\mathbb{C}^3)$ we have $\left(\begin{smallmatrix}\boldsymbol{0}\\ \boldsymbol{H}\end{smallmatrix}\right) \in \operatorname{dom}\mathring{\mathfrak{A}}_1$ and
$$\left\langle \pi_2\mathring{\mathfrak{A}}_1^*\left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right), \boldsymbol{H}\right\rangle_{\mathcal{X}_1} = \left\langle \mathring{\mathfrak{A}}_1^*\left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right), \left(\begin{smallmatrix}\boldsymbol{0}\\ \boldsymbol{H}\end{smallmatrix}\right)\right\rangle_{\mathcal{X}_1} \overset{(24)}{=} \langle \boldsymbol{E}, \operatorname{rot}\boldsymbol{H}\rangle_{\mathrm{L}^2(\Omega;\mathbb{C}^3)}, \tag{26}$$
where π_2 is the projection on the second component (the $\mathrm{L}^2(\Omega;\mathbb{C}^3)$ component). Hence, $\boldsymbol{E} \in \mathrm{H}(\operatorname{rot},\Omega)$ and $\pi_2\mathring{\mathfrak{A}}_2^*\left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right) = \operatorname{rot}\boldsymbol{E}$.

We conclude

$$\mathring{\mathfrak{A}}_1^*\begin{pmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{pmatrix} = \begin{pmatrix}\frac{\mathrm{d}}{\mathrm{d}\eta}\boldsymbol{V}\\ \operatorname{rot}\boldsymbol{E}\end{pmatrix} = \begin{bmatrix}\frac{\mathrm{d}}{\mathrm{d}\eta} & 0\\ 0 & \operatorname{rot}\end{bmatrix}\begin{pmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{pmatrix} \quad \text{for} \quad \begin{pmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{pmatrix} \in \operatorname{dom}\mathring{\mathfrak{A}}_1^*.$$

Now use the notation from (15) for the trace operators. By the boundary conditions of $\mathring{\mathfrak{A}}_1$ we have for $\left(\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{smallmatrix}\right) \in \operatorname{dom}\mathring{\mathfrak{A}}_1$ and $\left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right) \in \operatorname{dom}\mathring{\mathfrak{A}}_1^*$

$$\begin{aligned}
\left\langle \left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right), \mathring{\mathfrak{A}}_1\left(\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{smallmatrix}\right)\right\rangle_{\mathcal{X}_1} &= \left\langle \left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right), \left(\begin{smallmatrix}-\frac{\mathrm{d}}{\mathrm{d}\eta}\boldsymbol{I}+\frac{\mathrm{d}}{\mathrm{d}\eta}P_{\mathrm{el}}^*\gamma_\tau\boldsymbol{H}\\ \operatorname{rot}\boldsymbol{H}\end{smallmatrix}\right)\right\rangle_{\mathcal{X}_1}\\
&= \left\langle \tfrac{\mathrm{d}}{\mathrm{d}\eta}\boldsymbol{V}, \boldsymbol{I}\right\rangle_{\mathrm{L}^2((0,1);\mathbb{C}^k)} - \left\langle \tfrac{\mathrm{d}}{\mathrm{d}\eta}\boldsymbol{V}, P_{\mathrm{el}}^*\gamma_\tau\boldsymbol{H}\right\rangle_{\mathrm{L}^2((0,1);\mathbb{C}^k)}\\
&\qquad + \langle \operatorname{rot}\boldsymbol{E}, \boldsymbol{H}\rangle_{\mathrm{L}^2(\Omega;\mathbb{C}^3)} + \langle \pi_\tau\boldsymbol{E}, \gamma_\tau\boldsymbol{H}\rangle_{\mathcal{V}_\tau,\mathcal{V}_\tau^\times}\\
&= \left\langle \mathring{\mathfrak{A}}_1^*\left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right), \left(\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{smallmatrix}\right)\right\rangle_{\mathcal{X}_1} - \left[\langle P_{\mathrm{el}}\tfrac{\mathrm{d}}{\mathrm{d}\eta}\boldsymbol{V}, \gamma_\tau\boldsymbol{H}\rangle_{\mathrm{L}^2(\Gamma_{\mathrm{lat}})} - \langle \pi_\tau\boldsymbol{E}, \gamma_\tau\boldsymbol{H}\rangle_{\mathcal{V}_\tau,\mathcal{V}_\tau^\times}\right],
\end{aligned}$$

which implies $\langle \pi_\tau\boldsymbol{E}, \gamma_\tau\boldsymbol{H}\rangle_{\mathcal{V}_\tau,\mathcal{V}_\tau^\times} = \langle P_{\mathrm{el}}\frac{\mathrm{d}}{\mathrm{d}\eta}\boldsymbol{V}, \gamma_\tau\boldsymbol{H}\rangle_{\mathrm{L}^2(\Gamma_{\mathrm{lat}})}$ for all $\boldsymbol{H} \in \mathrm{C}^\infty(\overline{\Omega};\mathbb{C}^3)$. Therefore, $\pi_\tau\boldsymbol{E} = P_{\mathrm{el}}\frac{\mathrm{d}}{\mathrm{d}\eta}\boldsymbol{V}$ and $\left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right) \in \operatorname{dom}\mathfrak{A}_2$ and in turn $\mathring{\mathfrak{A}}_1^* \subseteq -\mathfrak{A}_2$. On the other hand it is straightforward to show $-\mathfrak{A}_2 \subseteq \mathring{\mathfrak{A}}_1^*$, which proves $\mathring{\mathfrak{A}}_1^* = -\mathfrak{A}_2$.

In order to show $\mathfrak{A}_1^* = -\mathring{\mathfrak{A}}_2$ we regard $\left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right) \in \operatorname{dom}\mathfrak{A}_1^*$ and the equation

$$\left\langle \mathfrak{A}_1^*\left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right), \left(\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{smallmatrix}\right)\right\rangle_{\mathcal{X}_1} = \left\langle \left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right), \mathfrak{A}_1\left(\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{smallmatrix}\right)\right\rangle_{\mathcal{X}_1} \quad \text{for all} \quad \left(\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{smallmatrix}\right) \in \operatorname{dom}\mathfrak{A}_1.$$

We can repeat the arguments (25) and (26) of the first part of the proof to conclude $\boldsymbol{V} \in \mathrm{H}^1((0,1);\mathbb{C}^k)$ and $\boldsymbol{E} \in \mathrm{H}(\operatorname{rot},\Omega)$. Moreover, since we can choose $\boldsymbol{I} \in \mathrm{H}^1((0,1);\mathbb{C}^k)$ in (25) we even obtain $\boldsymbol{V} \in \mathring{\mathrm{H}}^1((0,1);\mathbb{C}^k)$. Similarly, choosing

$\boldsymbol{H} \in \mathring{\mathrm{C}}^{\infty}_{\Gamma_{\mathrm{lat}}}(\Omega;\mathbb{C}^3)$ implies $\boldsymbol{E} \in \mathring{\mathrm{H}}_{\Gamma_{\mathrm{r}}}(\mathrm{rot},\Omega)$. Again repeating the remaining steps of the first part of the proof yields $\mathfrak{A}_1^* = -\mathring{\mathfrak{A}}_2$. ❑

Corollary 4.3. *Suppose that the spatial domains are as in Section 2.1. Then the operators as in (21a) and (22) fulfill*

(i) $\mathring{\mathfrak{A}}_2$ *and* $\mathfrak{A}_2$ *are closed,*
(ii) $\mathring{\mathfrak{A}}_1$ *and* $\mathfrak{A}_1$ *are closable,*
(iii) *the closures of* $\mathring{\mathfrak{A}}_1$ *and* $\mathfrak{A}_1$ *fulfill*

$$\overline{\mathring{\mathfrak{A}}_1} = -\mathfrak{A}_2^* \quad \textit{and} \quad \overline{\mathfrak{A}_1} = -\mathring{\mathfrak{A}}_2^*.$$

Proof. By Lemma 4.2 $\mathfrak{A}_2 = -\mathring{\mathfrak{A}}_1^*$ and $\mathring{\mathfrak{A}}_2 = -\mathfrak{A}_1^*$, which immediately implies the closedness of these operators. Since $\mathring{\mathfrak{A}}_2$ and $\mathfrak{A}_2$ are densely defined, we conclude that $\mathring{\mathfrak{A}}_2^*$ and $\mathfrak{A}_2^*$ are well-defined and closed operators. Hence,

$$\overline{\mathring{\mathfrak{A}}_1} = \mathring{\mathfrak{A}}_1^{**} = -\mathfrak{A}_2^* \quad \text{and} \quad \overline{\mathfrak{A}_1} = \mathfrak{A}_1^{**} = -\mathring{\mathfrak{A}}_2^*$$

implies that $\mathring{\mathfrak{A}}_1$ and $\mathfrak{A}_1$ are closable. ❑

Lemma 4.4. *Suppose that the spatial domains are as in Section 2.1, and let* $\mathfrak{A}_1$, $\mathfrak{A}_2$ *be as in (21a). Then, for all* $\left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right) \in \operatorname{dom}\mathfrak{A}_2$ *and* $\left(\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{smallmatrix}\right) \in \operatorname{dom}\mathfrak{A}_1$*, it holds that*

$$\left\langle \mathfrak{A}_2 \begin{pmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{pmatrix}, \begin{pmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{pmatrix}\right\rangle_{\mathcal{X}_1} + \left\langle \begin{pmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{pmatrix}, \mathfrak{A}_1 \begin{pmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{pmatrix}\right\rangle_{\mathcal{X}_1} = \left\langle \begin{pmatrix}\boldsymbol{V}(0)\\ -\boldsymbol{V}(1)\end{pmatrix}, \begin{pmatrix}\boldsymbol{I}_{\mathrm{tot}}(0)\\ \boldsymbol{I}_{\mathrm{tot}}(1)\end{pmatrix}\right\rangle_{\mathbb{C}^{2k}},$$

where $\boldsymbol{I}_{\mathrm{tot}} = \boldsymbol{I} - P_{\mathrm{el}}^*\gamma_\tau \boldsymbol{H}$.

Proof. Let $\left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right) \in \operatorname{dom}\mathfrak{A}_2$ and $\left(\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{smallmatrix}\right) \in \operatorname{dom}\mathfrak{A}_1$. Then

$$\left\langle \begin{bmatrix}-\frac{\mathrm{d}}{\mathrm{d}\eta} & 0\\ 0 & -\mathrm{rot}\end{bmatrix}\begin{pmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{pmatrix}, \begin{pmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{pmatrix}\right\rangle_{\mathcal{X}_1} + \left\langle \begin{pmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{pmatrix}, \begin{bmatrix}-\frac{\mathrm{d}}{\mathrm{d}\eta} & \frac{\mathrm{d}}{\mathrm{d}\eta}P_{\mathrm{el}}^*\gamma_\tau\\ 0 & \mathrm{rot}\end{bmatrix}\begin{pmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{pmatrix}\right\rangle_{\mathcal{X}_1}$$

$$\begin{aligned} &= \langle -\tfrac{\mathrm{d}}{\mathrm{d}\eta}\boldsymbol{V}, \boldsymbol{I}\rangle_{\mathrm{L}^2((0,1);\mathbb{C}^k)} + \langle \boldsymbol{V}, -\tfrac{\mathrm{d}}{\mathrm{d}\eta}(\boldsymbol{I} - P_{\mathrm{el}}^*\gamma_\tau\boldsymbol{H})\rangle_{\mathrm{L}^2((0,1);\mathbb{C}^k)} \\ &\quad + \underbrace{\langle -\mathrm{rot}\,\boldsymbol{E}, \boldsymbol{H}\rangle_{\mathrm{L}^2(\Omega;\mathbb{C}^3)} + \langle \boldsymbol{E}, \mathrm{rot}\,\boldsymbol{H}\rangle_{\mathrm{L}^2(\Omega;\mathbb{C}^3)}}_{=\langle \pi_\tau\boldsymbol{E}, \gamma_\tau\boldsymbol{H}\rangle_{\mathcal{V}_\tau,\mathcal{V}_\tau^\times}} \end{aligned}$$

Note that $\langle \pi_\tau\boldsymbol{E}, \gamma_\tau\boldsymbol{H}\rangle_{\mathcal{V}_\tau,\mathcal{V}_\tau^\times} = \langle \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{V}, P_{\mathrm{el}}^*\gamma_\tau\boldsymbol{H}\rangle_{\mathrm{L}^2((0,1);\mathbb{C}^k)}$ by the boundary condition of $\mathfrak{A}_2$. Hence, we further have

$$= \left\langle -\tfrac{\mathrm{d}}{\mathrm{d}\eta}\boldsymbol{V}, \boldsymbol{I} - P_{\mathrm{el}}^*\gamma_\tau\boldsymbol{H}\right\rangle_{\mathrm{L}^2((0,1);\mathbb{C}^k)} + \left\langle \boldsymbol{V}, -\tfrac{\mathrm{d}}{\mathrm{d}\eta}(\boldsymbol{I} - P_{\mathrm{el}}^*\gamma_\tau\boldsymbol{H})\right\rangle_{\mathrm{L}^2((0,1);\mathbb{C}^k)}$$

integration by parts formula for $\frac{\mathrm{d}}{\mathrm{d}\eta}$ finally yields

$$= -\big(\langle \boldsymbol{V}(1), \boldsymbol{I}_{\mathrm{tot}}(1)\rangle_{\mathbb{C}^k} - \langle \boldsymbol{V}(0), \boldsymbol{I}_{\mathrm{tot}}(0)\rangle_{\mathbb{C}^k}\big) = \left\langle \left(\begin{smallmatrix}\boldsymbol{V}(0)\\ -\boldsymbol{V}(1)\end{smallmatrix}\right), \left(\begin{smallmatrix}\boldsymbol{I}_{\mathrm{tot}}(0)\\ \boldsymbol{I}_{\mathrm{tot}}(1)\end{smallmatrix}\right)\right\rangle_{\mathbb{C}^{2k}},$$

which finishes the proof. ❑

Definition 4.5. Suppose that the spatial domains are as in Section 2.1, and let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be defined as in (21a). The *boundary mappings* $\hat{\mathfrak{B}}_1 \colon \operatorname{dom}\mathfrak{A}_1 \to \mathbb{C}^{2k}$ and $\hat{\mathfrak{B}}_2 \colon \operatorname{dom}\mathfrak{A}_2 \to \mathbb{C}^{2k}$ are specified by

$$\hat{\mathfrak{B}}_1 \begin{pmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{pmatrix} = \begin{pmatrix}\boldsymbol{I}_{\mathrm{tot}}(0)\\ \boldsymbol{I}_{\mathrm{tot}}(1)\end{pmatrix} \quad \text{and} \quad \hat{\mathfrak{B}}_2 \begin{pmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{pmatrix} = \begin{pmatrix}\boldsymbol{V}(0)\\ -\boldsymbol{V}(1)\end{pmatrix},$$

where $\boldsymbol{I}_{\mathrm{tot}} \coloneqq \boldsymbol{I} - P_{\mathrm{el}}^*\gamma_\tau\boldsymbol{H}$.

By Lemma 4.4 we have the abstract integration by parts formula

$$\langle \mathfrak{A}_2 x_2, x_1\rangle_{\mathcal{X}} + \langle x_2, \mathfrak{A}_1 x_1\rangle_{\mathcal{X}} = \langle \hat{\mathfrak{B}}_2 x_2, \hat{\mathfrak{B}}_1 x_1\rangle_{\mathbb{C}^{2k}},$$

where $x_1 = \left(\begin{smallmatrix}\boldsymbol{V}\\ \boldsymbol{E}\end{smallmatrix}\right) \in \operatorname{dom}\mathfrak{A}_1$ and $x_2 = \left(\begin{smallmatrix}\boldsymbol{I}\\ \boldsymbol{H}\end{smallmatrix}\right) \in \operatorname{dom}\mathfrak{A}_2$.

Lemma 4.6. *Under the preconditions in Definition 4.5, $\hat{\mathfrak{B}}_1$ and $\hat{\mathfrak{B}}_2$ are surjective.*

Proof. For given $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathbb{C}^{2k}$ we find a linear function $\boldsymbol{I}\colon [0,1] \to \mathbb{C}^k$ such that $\boldsymbol{I}(0) = a$ and $\boldsymbol{I}(1) = b$. Since $\boldsymbol{I}$ is a linear interpolation, we automatically have $\boldsymbol{I} \in \mathrm{H}^1((0,1);\mathbb{C}^k)$. We set $\boldsymbol{H} \coloneqq \boldsymbol{0}$, which gives $\left(\begin{smallmatrix} \boldsymbol{I} \\ \boldsymbol{H} \end{smallmatrix}\right) \in \operatorname{dom} \mathfrak{A}_1$ and

$$\hat{\mathfrak{B}}_1 \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{H} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}(0) - P_{\mathrm{el}}^* \gamma_\tau \boldsymbol{0} \\ \boldsymbol{I}(1) - P_{\mathrm{el}}^* \gamma_\tau \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Again by linear interpolation we can find a $\boldsymbol{V}\colon [0,1] \to \mathbb{C}^k$ such that $\boldsymbol{V}(0) = c$ and $-\boldsymbol{V}(1) = d$ for given $\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right) \in \mathbb{C}^{2k}$. Then we set $\boldsymbol{E} \coloneqq \hat{P}_{\mathrm{el}} \boldsymbol{V}$ so that $\left(\begin{smallmatrix} \boldsymbol{V} \\ \boldsymbol{E} \end{smallmatrix}\right) \in \operatorname{dom} \mathfrak{A}_2$ and

$$\hat{\mathfrak{B}}_2 \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{E} \end{pmatrix} = \begin{pmatrix} \boldsymbol{V}(0) \\ -\boldsymbol{V}(1) \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix},$$

which proves the assertion. ❑

Theorem 4.7. *Under the preconditions in Definition 4.5, $\|\cdot\|_{\operatorname{ran} \hat{\mathfrak{B}}_1}$ with*

$$\left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\|_{\operatorname{ran} \hat{\mathfrak{B}}_1} \coloneqq \inf \left\{ \left\| \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{H} \end{pmatrix} \right\|_{\operatorname{dom}(\mathfrak{A}_1)} \,\middle|\, \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_{\mathrm{tot}}(0) \\ \boldsymbol{I}_{\mathrm{tot}}(1) \end{pmatrix}, \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{H} \end{pmatrix} \in \operatorname{dom} \mathfrak{A}_1 \right\}$$

is a norm on $\mathbb{C}^{2k}$. In particular $\hat{\mathfrak{B}}_1\colon \operatorname{dom} \mathfrak{A}_1 \to \mathbb{C}^{2k}$ is continuous, where $\operatorname{dom} \mathfrak{A}_1$ is equipped with the graph norm of $\mathfrak{A}_1$.[2]

Note that also $\mathfrak{B}_2\colon \operatorname{dom} \mathfrak{A}_2 \to \mathbb{C}^{2k}$ is continuous, but we do not use this fact as $\mathfrak{A}_2$ is already closed.

Proof. The property $\|\lambda \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\|_{\operatorname{ran} \hat{\mathfrak{B}}_1} = |\lambda| \|\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\|_{\operatorname{ran} \hat{\mathfrak{B}}_1}$ is obvious and also the triangle inequality can be shown by applying the infimum on both sides of the triangle inequality for $\|\cdot\|_{\mathfrak{A}_1}$. These are well-known techniques, e.g., for factor spaces. The only tricky step is showing that $\|\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\|_{\operatorname{ran} \hat{\mathfrak{B}}_1} = 0$ implies $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = 0$.

Hence, let $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathbb{C}^{2k}$ be such that $\|\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\|_{\operatorname{ran} \hat{\mathfrak{B}}_1} = 0$. Then there exists a sequence $\left(\left(\begin{smallmatrix} \boldsymbol{I}_n \\ \boldsymbol{H}_n \end{smallmatrix}\right)\right)_{n\in\mathbb{N}}$ in $\operatorname{dom} \mathfrak{A}_1$ such that $\left(\begin{smallmatrix} \boldsymbol{I}_n \\ \boldsymbol{H}_n \end{smallmatrix}\right) \to 0$ w.r.t. $\|\cdot\|_{\mathfrak{A}_1}$ and $\left(\begin{smallmatrix} \boldsymbol{I}_{\mathrm{tot},n}(0) \\ \boldsymbol{I}_{\mathrm{tot},n}(1) \end{smallmatrix}\right) = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ for all $n \in \mathbb{N}$. Since also $\hat{\mathfrak{B}}_2$ is surjective, there exists $\left(\begin{smallmatrix} \boldsymbol{V} \\ \boldsymbol{E} \end{smallmatrix}\right) \in \operatorname{dom} \mathfrak{A}_2$ such that $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} \boldsymbol{V}(0) \\ -\boldsymbol{V}(1) \end{smallmatrix}\right)$. Therefore,

$$\begin{aligned} \left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\|_{\mathbb{C}^{2m}} &= \left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle_{\mathbb{C}^{2k}} = \left\langle \begin{pmatrix} \boldsymbol{V}(0) \\ -\boldsymbol{V}(1) \end{pmatrix}, \begin{pmatrix} \boldsymbol{I}_{\mathrm{tot},n}(0) \\ \boldsymbol{I}_{\mathrm{tot},n}(1) \end{pmatrix} \right\rangle_{\mathbb{C}^{2k}} \\ &= \left\langle \mathfrak{A}_2 \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{E}_n \end{pmatrix}, \begin{pmatrix} \boldsymbol{I}_n \\ \boldsymbol{H}_n \end{pmatrix} \right\rangle_{\mathcal{X}_1} + \left\langle \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{E} \end{pmatrix}, \mathfrak{A}_1 \begin{pmatrix} \boldsymbol{I}_n \\ \boldsymbol{H}_n \end{pmatrix} \right\rangle_{\mathcal{X}_1} \to 0, \end{aligned}$$

which implies $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = 0$.

Since we have shown that $\|\cdot\|_{\operatorname{ran} \hat{\mathfrak{B}}_1}$ is a norm on $\mathbb{C}^{2k}$ we know that it is equivalent to the standard norm on $\mathbb{C}^{2k}$. By construction $\hat{\mathfrak{B}}_1$ is continuous w.r.t. $\|\cdot\|_{\operatorname{ran} \hat{\mathfrak{B}}_2}$ and therefore also w.r.t. the standard norm on $\mathbb{C}^{2k}$. ❑

Since $\hat{\mathfrak{B}}_1\colon \operatorname{dom} \mathfrak{A}_1 \to \mathbb{C}^{2k}$ is continuous we can continuously extend $\hat{\mathfrak{B}}_1$ to $\operatorname{dom} \overline{\mathfrak{A}_1}$, where $\overline{\mathfrak{A}_1}$ is the operator closure of $\mathfrak{A}_1$. We will denote the extension of $\hat{\mathfrak{B}}_1$ still by $\hat{\mathfrak{B}}_1$. We immediately get the following corollary from Lemma 4.4.

[2] $\mathbb{C}^{2k}$ can be equipped with any norm as all of them are equivalent.

Corollary 4.8. *Suppose that the spatial domains are as in Section 2.1. Further, let* $\mathfrak{A}_1$, $\mathfrak{A}_2$ *be defined as in (21a), and let* $\hat{\mathfrak{B}}_1$, $\hat{\mathfrak{B}}_2$ *be as in Definition 4.5. Then, for* $\left(\begin{smallmatrix}\boldsymbol{I}\\\boldsymbol{H}\end{smallmatrix}\right) \in \operatorname{dom}\overline{\mathfrak{A}_1}$ *and* $\left(\begin{smallmatrix}\boldsymbol{V}\\\boldsymbol{E}\end{smallmatrix}\right) \in \operatorname{dom}\mathfrak{A}_2$ *we have*

$$\begin{aligned}\langle \mathfrak{A}_2\left(\begin{smallmatrix}\boldsymbol{V}\\\boldsymbol{E}\end{smallmatrix}\right), \left(\begin{smallmatrix}\boldsymbol{I}\\\boldsymbol{H}\end{smallmatrix}\right)\rangle_{\mathcal{X}_1} + \langle \left(\begin{smallmatrix}\boldsymbol{V}\\\boldsymbol{E}\end{smallmatrix}\right), \overline{\mathfrak{A}_1}\left(\begin{smallmatrix}\boldsymbol{I}\\\boldsymbol{H}\end{smallmatrix}\right)\rangle_{\mathcal{X}_1} &= \left\langle \hat{\mathfrak{B}}_2\left(\begin{smallmatrix}\boldsymbol{V}\\\boldsymbol{E}\end{smallmatrix}\right), \hat{\mathfrak{B}}_1\left(\begin{smallmatrix}\boldsymbol{I}\\\boldsymbol{H}\end{smallmatrix}\right)\right\rangle_{\mathbb{C}^{2k}} \\ &= \left\langle \begin{pmatrix}\boldsymbol{V}(0)\\-\boldsymbol{V}(1)\end{pmatrix}, \begin{pmatrix}\boldsymbol{I}_{\mathrm{tot}}(0)\\\boldsymbol{I}_{\mathrm{tot}}(1)\end{pmatrix}\right\rangle_{\mathbb{C}^{2k}}.\end{aligned}$$

Note that this abstract integration by parts formula gives rise to an abstract Green identity for the corresponding block operator $\overline{\mathfrak{J}} = \left[\begin{smallmatrix}0 & \mathfrak{A}_2\\ \overline{\mathfrak{A}_1} & 0\end{smallmatrix}\right]$. In particular, we obtain a boundary triple for $\overline{\mathfrak{J}}$.

Theorem 4.9. *Suppose that the spatial domains are as in Section 2.1. Further, let* $\mathcal{X}$ *and* $\mathfrak{J}$ *be defined as in (20b) and (21). Further, let* $\hat{\mathfrak{B}}_1$, $\hat{\mathfrak{B}}_2$ *be as in Definition 4.5.*

For $e^i = (\boldsymbol{I}^i, \boldsymbol{H}^i, \boldsymbol{V}^i, \boldsymbol{E}^i) \in \operatorname{dom}\overline{\mathfrak{J}}$, $i = 1, 2$, *we have*

$$\begin{aligned}\langle \overline{\mathfrak{J}}e^1, e^2\rangle_{\mathcal{X}} + \langle e^1, \overline{\mathfrak{J}}e^2\rangle_{\mathcal{X}} &= \left\langle \hat{\mathfrak{B}}_1\begin{pmatrix}\boldsymbol{I}^1\\\boldsymbol{H}^1\end{pmatrix}, \hat{\mathfrak{B}}_2\begin{pmatrix}\boldsymbol{V}^2\\\boldsymbol{E}^2\end{pmatrix}\right\rangle_{\mathbb{C}^{2k}} + \left\langle \hat{\mathfrak{B}}_2\begin{pmatrix}\boldsymbol{V}^1\\\boldsymbol{E}^1\end{pmatrix}, \hat{\mathfrak{B}}_1\begin{pmatrix}\boldsymbol{I}^2\\\boldsymbol{H}^2\end{pmatrix}\right\rangle_{\mathbb{C}^{2k}} \\ &= \left\langle \begin{pmatrix}\boldsymbol{I}^1_{\mathrm{tot}}(0)\\\boldsymbol{I}^1_{\mathrm{tot}}(1)\end{pmatrix}, \begin{pmatrix}\boldsymbol{V}^2(0)\\-\boldsymbol{V}^2(1)\end{pmatrix}\right\rangle_{\mathbb{C}^{2k}} + \left\langle \begin{pmatrix}\boldsymbol{V}^1(0)\\-\boldsymbol{V}^1(1)\end{pmatrix}, \begin{pmatrix}\boldsymbol{I}^2_{\mathrm{tot}}(0)\\\boldsymbol{I}^2_{\mathrm{tot}}(1)\end{pmatrix}\right\rangle_{\mathbb{C}^{2k}}.\end{aligned}$$

Proof. By applying Corollary 4.8 twice we obtain

$$\begin{aligned}\langle \overline{\mathfrak{J}}e^1, e^2\rangle_{\mathcal{X}} + \langle e^1, \overline{\mathfrak{J}}e^2\rangle_{\mathcal{X}} &= \left\langle \begin{pmatrix}0 & \mathfrak{A}_2\\ \overline{\mathfrak{A}_1} & 0\end{pmatrix}e^1, e^2\right\rangle_{\mathcal{X}} + \left\langle e^1, \begin{pmatrix}0 & \mathfrak{A}_2\\ \overline{\mathfrak{A}_1} & 0\end{pmatrix}e^2\right\rangle_{\mathcal{X}} \\ &= \left\langle \overline{\mathfrak{A}_1}\begin{pmatrix}\boldsymbol{I}^1\\\boldsymbol{E}^1\end{pmatrix}, \begin{pmatrix}\boldsymbol{V}^2\\\boldsymbol{H}^2\end{pmatrix}\right\rangle_{\mathcal{X}_1} + \left\langle \begin{pmatrix}\boldsymbol{I}^1\\\boldsymbol{E}^1\end{pmatrix}, \mathfrak{A}_2\begin{pmatrix}\boldsymbol{V}^2\\\boldsymbol{H}^2\end{pmatrix}\right\rangle_{\mathcal{X}_1} \\ &\quad + \left\langle \mathfrak{A}_2\begin{pmatrix}\boldsymbol{V}^1\\\boldsymbol{H}^1\end{pmatrix}, \begin{pmatrix}\boldsymbol{I}^2\\\boldsymbol{E}^2\end{pmatrix}\right\rangle_{\mathcal{X}_1} + \left\langle \begin{pmatrix}\boldsymbol{V}^1\\\boldsymbol{H}^1\end{pmatrix}, \overline{\mathfrak{A}_1}\begin{pmatrix}\boldsymbol{I}^2\\\boldsymbol{E}^2\end{pmatrix}\right\rangle_{\mathcal{X}_1} \\ &= \left\langle \hat{\mathfrak{B}}_1\begin{pmatrix}\boldsymbol{I}^1\\\boldsymbol{H}^1\end{pmatrix}, \hat{\mathfrak{B}}_2\begin{pmatrix}\boldsymbol{V}^2\\\boldsymbol{E}^2\end{pmatrix}\right\rangle_{\mathbb{C}^{2k}} + \left\langle \hat{\mathfrak{B}}_2\begin{pmatrix}\boldsymbol{V}^1\\\boldsymbol{E}^2\end{pmatrix}, \hat{\mathfrak{B}}_1\begin{pmatrix}\boldsymbol{I}^2\\\boldsymbol{H}^2\end{pmatrix}\right\rangle_{\mathbb{C}^{2k}},\end{aligned}$$

which proves the assertion. ❑

Corresponding to $\hat{\mathfrak{B}}_1$ and $\hat{\mathfrak{B}}_2$ we define the operators on $\operatorname{dom}\overline{\mathfrak{J}}$ by projections. In particular for $e = (\boldsymbol{I}, \boldsymbol{H}, \boldsymbol{V}, \boldsymbol{E}) \in \operatorname{dom}\overline{\mathfrak{J}}$ we define

$$\begin{aligned}\mathfrak{B}_1 e = \mathfrak{B}_1\begin{pmatrix}\boldsymbol{I}\\\boldsymbol{H}\\\boldsymbol{V}\\\boldsymbol{E}\end{pmatrix} &\coloneqq \hat{\mathfrak{B}}_1\left(\begin{smallmatrix}\boldsymbol{I}\\\boldsymbol{H}\end{smallmatrix}\right) = \begin{pmatrix}\boldsymbol{I}_{\mathrm{tot}}(0)\\\boldsymbol{I}_{\mathrm{tot}}(1)\end{pmatrix}, \\ \mathfrak{B}_2 e = \mathfrak{B}_2\begin{pmatrix}\boldsymbol{I}\\\boldsymbol{H}\\\boldsymbol{V}\\\boldsymbol{E}\end{pmatrix} &\coloneqq \hat{\mathfrak{B}}_2\left(\begin{smallmatrix}\boldsymbol{V}\\\boldsymbol{E}\end{smallmatrix}\right) = \begin{pmatrix}\boldsymbol{V}(0)\\-\boldsymbol{V}(1)\end{pmatrix}.\end{aligned} \tag{27}$$

This allows us to write the abstract Green identity from Theorem 4.9 just as

$$\langle \overline{\mathfrak{J}}e_1, e_2\rangle_{\mathcal{X}} + \langle e_1, \overline{\mathfrak{J}}e_2\rangle_{\mathcal{X}} = \langle \mathfrak{B}_1 e_1, \mathfrak{B}_2 e_2\rangle_{\mathbb{C}^{2k}} + \langle \mathfrak{B}_2 e_1, \mathfrak{B}_1 e_2\rangle_{\mathbb{C}^{2k}} \tag{28}$$

Corollary 4.10. *Suppose that the spatial domains are as in Section 2.1. Further, let* $\mathcal{X}$ *and* $\mathfrak{J}$ *be defined as in (20b) and (21), and let* $\mathfrak{B}_1$, $\mathfrak{B}_2$ *be as in (27). Then* $(\mathbb{C}^{2k}, \mathfrak{B}_1, \mathfrak{B}_2)$ *is a boundary triple (see Definition A.2) for* $\overline{\mathfrak{J}}$, *i.e., the following properties are fulfilled.*

(i) $-\overline{\mathfrak{J}}^* \subseteq \overline{\mathfrak{J}}$,

(ii) *The abstract Green identity (28) is satisfied for all* $e_1, e_2 \in \operatorname{dom}\overline{\mathfrak{J}}$ *and*

(iii) $\left[\begin{smallmatrix}\mathfrak{B}_1\\\mathfrak{B}_2\end{smallmatrix}\right] \colon \operatorname{dom}\overline{\mathfrak{J}} \to \left[\begin{smallmatrix}\mathbb{C}^{2k}\\\mathbb{C}^{2k}\end{smallmatrix}\right]_{\times}$ *is surjective and bounded.*

Proof. We prove these three statements in the given order.

(i) Note that $\mathfrak{J} = \left[\begin{smallmatrix}0 & \mathfrak{A}_2\\ \mathfrak{A}_1 & 0\end{smallmatrix}\right]$ and therefore $\overline{\mathfrak{J}} = \left[\begin{smallmatrix}0 & \mathfrak{A}_2\\ \overline{\mathfrak{A}_1} & 0\end{smallmatrix}\right]$. Hence, by (23) and Lemma 4.2 follows

$$\overline{\mathfrak{J}}^* = \mathfrak{J}^* = \begin{bmatrix}0 & \mathfrak{A}_2\\ \mathfrak{A}_1 & 0\end{bmatrix}^* = \begin{bmatrix}0 & \mathfrak{A}_1^*\\ \mathfrak{A}_2^* & 0\end{bmatrix} = \begin{bmatrix}0 & -\mathring{\mathfrak{A}}_2\\ -\overline{\mathring{\mathfrak{A}}_1} & 0\end{bmatrix} \subseteq -\begin{bmatrix}0 & \mathfrak{A}_2\\ \overline{\mathfrak{A}_1} & 0\end{bmatrix} = -\overline{\mathfrak{J}}.$$

(ii) The abstract Green identity follows from the definition of $\mathfrak{B}_1$ and $\mathfrak{B}_2$, and Theorem 4.9

(iii) Surjectivity follows from Lemma 4.6. Boundedness is a consequence of [Skr21a, Lem. 2.4.7]. ❑

In the following theorem we will parameterize boundary conditions for our coupled systems that are more or less the boundary conditions for the telegraphers equations in port-Hamiltonian modeling (in the sense of [JZ12]) alone. This means that the allegedly more difficult Maxwell system does not have a big impact for the parametrization of the boundary conditions, which is surprising.

Theorem 4.11. *Suppose that Assumption 2.2 is satisfied, and the spatial domains are as in Section 2.1. Further, let $\mathcal{X}$ and $\mathfrak{J}$ be defined as in (20b) and (21), and let $\mathfrak{B}_1$, $\mathfrak{B}_2$ be as in (27). Further, assume that $W_B \in \mathbb{C}^{2k\times 4k}$ has the property as in Assumption 2.2. Then the operator defined by*

$$\mathfrak{A}e = \overline{\mathfrak{J}}e = \begin{bmatrix} 0 & \overline{\mathfrak{A}_2} \\ \mathfrak{A}_1 & 0 \end{bmatrix} e \tag{29a}$$

with domain

$$\operatorname{dom}\mathfrak{A} = \left\{ e \in \operatorname{dom}\overline{\mathfrak{J}} \,\middle|\, W_B \begin{bmatrix} \mathfrak{B}_1 e \\ \mathfrak{B}_2 e \end{bmatrix} = 0 \right\} \tag{29b}$$

is maximally dissipative.

Proof. Partitioning $W_B = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$ with $W_1, W_2 \in \mathbb{C}^{2k\times 2k}$, we can equivalently write the boundary condition in $\operatorname{dom}\mathfrak{A}$ as

$$\begin{bmatrix} \mathfrak{B}_1 x \\ \mathfrak{B}_2 x \end{bmatrix} \in \ker \begin{bmatrix} W_1 & W_2 \end{bmatrix}$$

As W_1, W_2 satisfy the conditions of Lemma A.4 we have that $\ker \begin{bmatrix} W_1 & W_2 \end{bmatrix} = \ker W_B$ is a maximally dissipative linear relation. Now using that, by Corollary 4.10, $(\mathbb{C}^{2k}, \mathfrak{B}_1, \mathfrak{B}_2)$ is a boundary triple for $\overline{\mathfrak{J}}$, the result follows from Theorem A.3. ❑

The requirement for boundedness and positivity on $\boldsymbol{C}$, $\boldsymbol{L}$, $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$, as stated in Assumption 2.1 and 2.4, implies that the operator $\mathcal{H}\colon \mathcal{X} \to \mathcal{X}$, as defined in (20a), is strictly positive and bounded. Consequently, we define

$$\left\langle \begin{pmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{B}_1 \\ \boldsymbol{q}_1 \\ \boldsymbol{D}_1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\psi}_2 \\ \boldsymbol{B}_2 \\ \boldsymbol{q}_2 \\ \boldsymbol{D}_2 \end{pmatrix} \right\rangle_{\mathcal{H}} := \left\langle \begin{pmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{B}_1 \\ \boldsymbol{q}_1 \\ \boldsymbol{D}_1 \end{pmatrix}, \mathcal{H} \begin{pmatrix} \boldsymbol{\psi}_2 \\ \boldsymbol{B}_2 \\ \boldsymbol{q}_2 \\ \boldsymbol{D}_2 \end{pmatrix} \right\rangle_{\mathcal{X}} \tag{30}$$

which is an equivalent inner product on $\mathcal{X}$. We denote $\mathcal{X}$ endowed with $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ as $\mathcal{X}_{\mathcal{H}}$, i.e., the only difference between $\mathcal{X}$ and $\mathcal{X}_{\mathcal{H}}$ is the choice of the inner product, but the topology is the same. The corresponding norm $\|\cdot\|_{\mathcal{H}} = \sqrt{\langle\cdot,\cdot\rangle_{\mathcal{H}}}$ is also called the energy norm.

Theorem 4.12. *Suppose that Assumption 2.1, 2.2 and 2.4 are satisfied, and the spatial domains are as in Section 2.1. Further, let the operators and spaces $\mathfrak{J}$, $\mathfrak{R}$, $\mathcal{H}$ and $\mathcal{X}$ be defined as in (20) and (21), and let $\mathfrak{B}_1$, $\mathfrak{B}_2$ be as in (27). Then the operator F defined by the restriction of $\overline{\mathfrak{J}} - \mathfrak{R}$ to*

$$\operatorname{dom} F = \left\{ e \in \operatorname{dom}\overline{\mathfrak{J}} \,\middle|\, W_B \begin{pmatrix} \mathfrak{B}_1 e \\ \mathfrak{B}_2 e \end{pmatrix} = 0 \right\}$$

is maximally dissipative. Further, the following holds:

(a) *$F\mathcal{H}$ generates a strongly continuous semigroup on $\mathcal{X}$. This semigroup is contractive with respect to the norm $\|\cdot\|_{\mathcal{H}}$ as in (30).*

(b) *If* $\mathfrak{R} = 0$ *and*

$$W_B \begin{bmatrix} 0 & \mathrm{id}_{2k} \\ \mathrm{id}_{2k} & 0 \end{bmatrix} W_B^* = 0, \tag{31}$$

then F *is skew-adjoint. That is,* $F^* = -F$.

(c) *If* (31) *holds, then* $F\mathcal{H}$ *generates a strongly continuous group on* $\mathcal{X}$. *If, further,* $\mathfrak{R} = 0$, *then this group is unitary with respect to the norm* $\|\cdot\|_{\mathcal{H}}$ *as in* (30).

Proof. The operator $\mathfrak{A}$ as in (29) maximally dissipative by Theorem 4.11. By using the $\mathfrak{R}$ is bounded and dissipative, we can conclude that $F = \mathfrak{A} - \mathfrak{R}$ is maximally dissipative as well.

(a) Maximal dissipativity of F directly implies that $F\mathcal{H}$ is maximally dissipative when $\mathcal{X}$ is equipped with the norm $\|\cdot\|_{\mathcal{H}}$. Therefore, the Lumer-Phillips theorem [TW09, Prop. 3.8.4] yields that $F\mathcal{H}$ generates a contractive semigroup on $\mathcal{X}$ equipped with $\|\cdot\|_{\mathcal{H}}$. With the assistance of norm equivalence, this semigroup is also strongly continuous on $\mathcal{X}$ when equipped with the standard norm.

(b) Note that the condition (31) on $W_B = [\, W_1 \; W_2 \,]$ implies that $\ker W_B = \ker[\, W_1 \; W_2 \,]$ is skew-adjoint. Hence, this property passes on to F.

(c) If $\mathfrak{R} = 0$ and (31) are satisfied, we can use (b) to deduce that $F\mathcal{H}$ is skew-adjoint with respect to the norm $\|\cdot\|_{\mathcal{H}}$. Then, by [TW09, Thm. 3.8.6], $F\mathcal{H}$ generates a unitary group on $\mathcal{X}$ equipped with $\|\cdot\|_{\mathcal{H}}$. Norm equivalence yields that this group exhibits strong continuity on $\mathcal{X}$ equipped with the standard norm.
If $\mathfrak{R}$ is not necessarily the zero operator, we observe that $F\mathcal{H}$ represents a bounded perturbation of the generator of a C_0-group on $\mathcal{X}$. Utilizing [TW09, Thm. 2.11.2], we can assert that $F\mathcal{H}$ itself generates a C_0-group. ❑

5. System node representation and well-posedness

Here, we further explore the systems-theoretic analysis of the coupled field-cable system by examining its input-output characteristics. To accomplish this, we unitlize the system node framework established by Staffans in [Sta05]. Assume that $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ are Hilbert spaces. Our system is represented in the form

$$\dot{x}(t) = A\&B \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \quad y(t) = C\&D \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \tag{32}$$

where $A\&B\colon \mathrm{dom}(A\&B) \subset \mathcal{X} \times \mathcal{U} \to \mathcal{X}$, $C\&D\colon \mathrm{dom}(C\&D) \subset \mathcal{X} \times \mathcal{U} \to \mathcal{Y}$ are linear operators with specific properties detailed subsequently. Unlike the finite-dimensional case, the operators $A\&B$ and $C\&D$ do not separate into distinct components corresponding to the state and input. This is primarily motivated by the application of boundary control for partial differential equations.
The autonomous dynamics (i.e., those with trivial input $u = 0$) are determined by the *main operator* $A\colon \mathrm{dom}(A) \subset X \to X$ with $\mathrm{dom}(A) := \{x \in X \mid \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) \in \mathrm{dom}(A\&B)\}$ and $Ax := A\&B \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$ for all $x \in \mathrm{dom}(A)$. Next, we state the essential conditions on the operators $A\&B$ and $C\&D$ under which they constitute a system node.

Definition 5.1 (System node)**.** A *system node* on the triple $(\mathcal{X}, \mathcal{U}, \mathcal{Y})$ of Hilbert spaces is a linear operator $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ with $A\&B\colon \mathrm{dom}(A\&B) \subset \mathcal{X} \times \mathcal{U} \to \mathcal{X}$, $C\&D\colon \mathrm{dom}(C\&D) \subset \mathcal{X} \times \mathcal{U} \to \mathcal{Y}$ satisfying the following conditions:

(i) $A\&B$ is closed.
(ii) $C\&D \in \mathcal{L}_{\mathrm{b}}(\mathrm{dom}(A\&B), \mathcal{Y})$.
(iii) For all $u \in \mathcal{U}$, there exists some $x \in \mathcal{X}$ with $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathrm{dom}(S)$.
(iv) The main operator A is the generator of a strongly continuous semigroup $\mathfrak{A}(\cdot)\colon \mathbb{R}_{\geq 0} \to \mathcal{L}_{\mathrm{b}}(\mathcal{X})$ on $\mathcal{X}$.

Next, we define our solution concepts for (32).

Definition 5.2 (Classical/generalized trajectories)**.** Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on $(\mathcal{X},\mathcal{U},\mathcal{Y})$, and let $T \in \mathbb{R}_{>0}$.
A *classical trajectory* for (32) on $[0,T]$ is a triple

$$(x,u,y) \in \mathrm{C}^1([0,T];\mathcal{X}) \times \mathrm{C}([0,T];\mathcal{U}) \times \mathrm{C}([0,T];\mathcal{Y})$$

which for all $t \in [0,T]$ satisfies (32).
A *generalized trajectory* for (32) on $[0,T]$ is a limit of classical trajectories for (32) on $[0,T]$ in the topology of $\mathrm{C}([0,T];\mathcal{X}) \times \mathrm{L}^2([0,T];\mathcal{U}) \times \mathrm{L}^2([0,T];\mathcal{Y})$.

It is shown in [Sta05, Thm. 4.3.9] that, if $x_0 \in \mathcal{X}$ and $u \in W^{1,2}([0,T];\mathcal{U})$ with $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \operatorname{dom}(A\&B))$, then there exist unique $x \in \mathrm{C}([0,T];\mathcal{X})$ with $x(0) = x_0$ and $y \in \mathrm{L}^2([0,T];\mathcal{Y})$, such that (x,u,y) is a classical trajectory for (32).
Well-posed systems are those with the property that the output and state depend continuously on the initial state and input.

Definition 5.3 (Well-posed systems)**.** Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on $(\mathcal{X},\mathcal{U},\mathcal{Y})$. The system (32) is called *well-posed*, if for some (and hence all) $t > 0$, there exists some $c_t > 0$, such that the classical (and thus also the generalized) trajectories for (32) on $[0,t]$ fulfill

$$\|x(t)\|_{\mathcal{X}} + \|y\|_{\mathrm{L}^2([0,t];\mathcal{Y})} \leq c_t\big(\|x(0)\|_{\mathcal{X}} + \|u\|_{\mathrm{L}^2([0,t];\mathcal{U})}\big). \tag{33}$$

Now we define the system node corresponding to the coupled field-cable system. To this end, under Assumption 2.1, 2.2 and 2.4, we consider the spatial domains as specified in Section 2.1. Further, let $\mathfrak{J}$, $\mathfrak{R}$, $\mathcal{H}$ and $\mathcal{X}$ as in (20) and (21), and, additionally $\mathcal{U} = \mathbb{C}^m$, $\mathcal{Y} = \mathbb{C}^p$. Let $W_{C,\mathrm{out}} \in \mathbb{C}^{p\times 4k}$, and assume that $W_{B,\mathrm{inp}} \in \mathbb{C}^{m\times 4k}$, $W_{B,0} \in \mathbb{C}^{(2k-m)\times 4k}$, fulfill Assumption 2.2. The operators defining the system node are given by

$$F\&G\left[\begin{smallmatrix} \mathcal{H} & 0 \\ 0 & \mathrm{id}_m \end{smallmatrix}\right], \quad K\&L\left[\begin{smallmatrix} \mathcal{H} & 0 \\ 0 & \mathrm{id}_m \end{smallmatrix}\right] \tag{34a}$$

with

$$\operatorname{dom}(F\&G) = \left\{ \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{H} \\ \boldsymbol{V} \\ \boldsymbol{E} \\ u \end{pmatrix} \in \operatorname{dom}\overline{\mathfrak{J}} \times \mathbb{C}^m \,\middle|\, \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{bmatrix} W_{B,\mathrm{inp}} \\ W_{B,0} \end{bmatrix} \begin{pmatrix} \boldsymbol{V}(0) \\ \boldsymbol{I}(0) \\ \boldsymbol{V}(1) \\ -\boldsymbol{I}(1) \end{pmatrix} \right\}, \tag{34b}$$

$$F\&G\begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{H} \\ \boldsymbol{V} \\ \boldsymbol{E} \\ u \end{pmatrix} = (\mathfrak{J} - \mathfrak{R})\begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{H} \\ \boldsymbol{V} \\ \boldsymbol{E} \end{pmatrix}, \tag{34c}$$

$$K\&L\begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{H} \\ \boldsymbol{V} \\ \boldsymbol{E} \\ u \end{pmatrix} = W_{C,\mathrm{out}}\begin{pmatrix} \boldsymbol{V}(0) \\ \boldsymbol{I}(0) \\ \boldsymbol{V}(1) \\ -\boldsymbol{I}(1) \end{pmatrix}, \tag{34d}$$

Theorem 5.4. *Suppose that Assumption 2.1, 2.2 and 2.4 are satisfied, and the spatial domain is as depicted in Section 2.1. Then for $\mathcal{X}$ as in (20b), and $F\&G$, $K\&L$ as in (34), and $\mathcal{H}$ as in (20a), the operator*

$$S := \begin{bmatrix} F\&G \\ K\&L \end{bmatrix}\left[\begin{smallmatrix} \mathcal{H} & 0 \\ 0 & \mathrm{id}_m \end{smallmatrix}\right] \tag{35}$$

is a system node on $(\mathcal{X},\mathbb{C}^m,\mathbb{C}^p)$. Further, if the output is co-located, then

$$M = \begin{bmatrix} F\&G \\ -K\&L \end{bmatrix} \tag{36}$$

with domain $\operatorname{dom}(M) = \operatorname{dom}(F\&G)$ is a maximally dissipative operator. In this case, all generalized trajectories of the system corresponding to the system node S

on $[0,T]$ *fulfill, for* $\mathfrak{B}_1$, $\mathfrak{B}_2$ *as in* (27), *for all* $t \in [0,T]$

$$\begin{aligned}
&\tfrac{1}{2}\big\langle x(t), \mathcal{H}x(t)\big\rangle_{\mathcal{X}} - \tfrac{1}{2}\big\langle x(0), \mathcal{H}x(0)\big\rangle_{\mathcal{X}} \\
&\quad = \int_0^t \operatorname{Re}\langle u(\tau), y(\tau)\rangle_{\mathbb{C}^m}\,\mathrm{d}\tau + \int_0^t \operatorname{Re}\langle \mathcal{H}x(\tau), \mathfrak{R}\mathcal{H}x(\tau)\rangle_{\mathcal{X}}\,\mathrm{d}\tau \\
&\qquad + \tfrac{1}{2}\int_0^t \left\langle \left[\begin{smallmatrix}\mathfrak{B}_1\\ \mathfrak{B}_2\end{smallmatrix}\right]\mathcal{H}x(\tau), \left(\left[\begin{smallmatrix}0 & \mathrm{id}_{2k}\\ \mathrm{id}_{2k} & 0\end{smallmatrix}\right] - \left[\begin{smallmatrix}W_B\\ W_C\end{smallmatrix}\right]^* \left[\begin{smallmatrix}0 & \mathrm{id}_{2k}\\ \mathrm{id}_{2k} & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}W_B\\ W_C\end{smallmatrix}\right]\right)\left[\begin{smallmatrix}\mathfrak{B}_1\\ \mathfrak{B}_2\end{smallmatrix}\right]\mathcal{H}x(\tau)\right\rangle_{\mathbb{C}^{4k}} \mathrm{d}\tau\,.
\end{aligned} \tag{37}$$

Proof. To show that S is a system node on $(\mathcal{X}, \mathbb{C}^m, \mathbb{C}^p)$, we successively prove the four properties in Definition 5.1 in the order (iv), (iii), (i), (ii).
The semigroup property of $F\mathcal{H}$ (i.e., property (iv)) is established by Theorem 4.12, as the main operator of S is precisely the one addressed in that theorem.
Property (iii) is a direct consequence of Corollary 4.10 (iii).
To prove (i), let us assume that $(x_n, u_n)_{n\in\mathbb{N}}$ is a sequence in $\operatorname{dom}((F\mathcal{H})\&G)$ that converges to $(x, u) \in \mathcal{X} \times \mathbb{C}^m$ (in the $\mathcal{X} \times \mathbb{C}^m$ topology). Additionally, suppose that $(F\&G(\mathcal{H}x_n, u_n))_{n\in\mathbb{N}}$ converges in $\mathcal{X}$ to z.

With the definition of $F\&G$ and the boundedness of $\mathcal{H}$ and $\mathfrak{R}$, we can conclude that $(\mathcal{H}x_n)$ converges to $\mathcal{H}x$, and $(\overline{\mathfrak{J}}x_n)$ converges in $\mathcal{X}$ to $z + \mathfrak{R}\mathcal{H}x$. Now, since $\overline{\mathfrak{J}}$ is closed, we can assert that $\mathcal{H}x$ belongs to $\operatorname{dom}\overline{\mathfrak{J}}$, with $\mathfrak{J}\mathcal{H}x = z + \mathfrak{R}\mathcal{H}x$. This implies that $z = (\mathfrak{J} - \mathfrak{R})\mathcal{H}x$. In particular, $(\mathcal{H}x_n)$ converges in the topology of $\operatorname{dom}\mathfrak{J}$ to x. Partitioning

$$\mathcal{H}x_n = \begin{pmatrix}\boldsymbol{I}_n\\ \boldsymbol{H}_n\\ \boldsymbol{V}_n\\ \boldsymbol{E}_n\end{pmatrix}, \quad \mathcal{H}x = \begin{pmatrix}\boldsymbol{I}\\ \boldsymbol{H}\\ \boldsymbol{V}\\ \boldsymbol{E}\end{pmatrix}, \tag{38}$$

we can conclude from Corollary 4.10 that

$$\left(\begin{smallmatrix}u_n\\ 0\end{smallmatrix}\right) = \begin{bmatrix}W_{B,\mathrm{inp}}\\ W_{B,0}\end{bmatrix}\begin{pmatrix}\boldsymbol{V}_n(0)\\ \boldsymbol{I}_n(0)\\ \boldsymbol{V}_n(1)\\ -\boldsymbol{I}_n(1)\end{pmatrix} \underset{n\to\infty}{\to} \begin{bmatrix}W_{B,\mathrm{inp}}\\ W_{B,0}\end{bmatrix}\begin{pmatrix}\boldsymbol{V}(0)\\ \boldsymbol{I}(0)\\ \boldsymbol{V}(1)\\ -\boldsymbol{I}(1)\end{pmatrix} = \left(\begin{smallmatrix}u\\ 0\end{smallmatrix}\right).$$

The definition of $F\&G$ now yields that $(\mathcal{H}x, u) \in \operatorname{dom}(F\&G)$ with

$$F\&G\left(\begin{smallmatrix}\mathcal{H}x\\ u\end{smallmatrix}\right) = (\mathfrak{J} - \mathfrak{R})\mathcal{H}x = z.$$

Hence we have shown that $F\&G$ is closed.
Property (ii) holds as, according to Corollary 4.10,(iii), the evaluation at the boundary of the transmission line represents a bounded operator on $\operatorname{dom}(\overline{\mathfrak{J}})$.
To complete the proof of the result, we now assume that the output is co-located, i.e., (7) holds for some $W_C \in \mathbb{C}^{2k\times 4k}$ with (8). Let $\left(\begin{smallmatrix}e\\ u\end{smallmatrix}\right) \in \operatorname{dom}(F\&G)$, and partition e as $\mathcal{H}x$ in (38). Then

$$\operatorname{Re}\langle \mathfrak{B}_1 e, \mathfrak{B}_2 e\rangle_{\mathbb{C}^{2k}} = \tfrac{1}{2}\left\langle \left[\begin{smallmatrix}\mathfrak{B}_1\\ \mathfrak{B}_2\end{smallmatrix}\right]e, \left[\begin{smallmatrix}0 & \mathrm{id}_{2k}\\ \mathrm{id}_{2k} & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}\mathfrak{B}_1\\ \mathfrak{B}_2\end{smallmatrix}\right]e\right\rangle_{\mathbb{C}^{4k}}$$

and

$$\begin{aligned}
\operatorname{Re}\left\langle u, W_{C,\mathrm{out}}\left[\begin{smallmatrix}\mathfrak{B}_1\\ \mathfrak{B}_2\end{smallmatrix}\right]e\right\rangle_{\mathbb{C}^m} &= \operatorname{Re}\left\langle \left(\begin{smallmatrix}u\\ 0\end{smallmatrix}\right), W_C\left[\begin{smallmatrix}\mathfrak{B}_1\\ \mathfrak{B}_2\end{smallmatrix}\right]e\right\rangle_{\mathbb{C}^{2k}} \\
&= \operatorname{Re}\left\langle W_B\left[\begin{smallmatrix}\mathfrak{B}_1\\ \mathfrak{B}_2\end{smallmatrix}\right]e, W_C\left[\begin{smallmatrix}\mathfrak{B}_1\\ \mathfrak{B}_2\end{smallmatrix}\right]e\right\rangle_{\mathbb{C}^{2k}} \\
&= \tfrac{1}{2}\left\langle \left[\begin{smallmatrix}W_B\\ W_C\end{smallmatrix}\right]\left[\begin{smallmatrix}\mathfrak{B}_1\\ \mathfrak{B}_2\end{smallmatrix}\right]e, \left[\begin{smallmatrix}0 & \mathrm{id}_{2k}\\ \mathrm{id}_{2k} & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}W_B\\ W_C\end{smallmatrix}\right]\left[\begin{smallmatrix}\mathfrak{B}_1\\ \mathfrak{B}_2\end{smallmatrix}\right]e\right\rangle_{\mathbb{C}^{4k}} \\
&= \tfrac{1}{2}\left\langle \left[\begin{smallmatrix}\mathfrak{B}_1\\ \mathfrak{B}_2\end{smallmatrix}\right]e, \left[\begin{smallmatrix}W_B\\ W_C\end{smallmatrix}\right]^*\left[\begin{smallmatrix}0 & \mathrm{id}_{2k}\\ \mathrm{id}_{2k} & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}W_B\\ W_C\end{smallmatrix}\right]\left[\begin{smallmatrix}\mathfrak{B}_1\\ \mathfrak{B}_2\end{smallmatrix}\right]e\right\rangle_{\mathbb{C}^{4k}}.
\end{aligned}$$

Using these two identities, we obtain that

$$\begin{aligned}
&\mathrm{Re}\Big\langle \left(\begin{smallmatrix} e \\ u \end{smallmatrix}\right), \left[\begin{smallmatrix} F\&G \\ -K\&L \end{smallmatrix}\right]\left(\begin{smallmatrix} e \\ u \end{smallmatrix}\right)\Big\rangle_{\mathcal{X}\times\mathbb{C}^m} \\
&\quad = \mathrm{Re}\,\langle e, F\&G\left(\begin{smallmatrix} e \\ u \end{smallmatrix}\right)\rangle_{\mathcal{X}} - \mathrm{Re}\,\langle u, K\&L\left(\begin{smallmatrix} e \\ u \end{smallmatrix}\right)\rangle_{\mathbb{C}^m} \\
&\quad = \mathrm{Re}\,\langle e, \mathfrak{J}e\rangle_{\mathcal{X}} - \mathrm{Re}\,\langle e, \mathfrak{R}e\rangle_{\mathcal{X}} - \mathrm{Re}\,\big\langle u, W_{C,\mathrm{out}}\left[\begin{smallmatrix} \mathfrak{B}_1 \\ \mathfrak{B}_2 \end{smallmatrix}\right]\big\rangle_{\mathbb{C}^m} \\
&\quad \overset{(28)}{=} \mathrm{Re}\langle \mathfrak{B}_1 e, \mathfrak{B}_2 e\rangle_{\mathbb{C}^{2m}} - \mathrm{Re}\,\langle e, \mathfrak{R}e\rangle_{\mathcal{X}} - \mathrm{Re}\,\big\langle u, W_{C,\mathrm{out}}\left[\begin{smallmatrix} \mathfrak{B}_1 \\ \mathfrak{B}_2 \end{smallmatrix}\right]\big\rangle_{\mathbb{C}^m} \\
&\quad = -\,\mathrm{Re}\,\langle e, \mathfrak{R}e\rangle_{\mathcal{X}} + \mathrm{Re}\langle \mathfrak{B}_1 e, \mathfrak{B}_2 e\rangle_{\mathbb{C}^{2m}} - \mathrm{Re}\,\big\langle u, W_{C,\mathrm{out}}\left[\begin{smallmatrix} \mathfrak{B}_1 \\ \mathfrak{B}_2 \end{smallmatrix}\right]e\big\rangle_{\mathbb{C}^m} \\
&\quad = -\,\mathrm{Re}\,\langle e, \mathfrak{R}e\rangle_{\mathcal{X}} \\
&\qquad + \tfrac{1}{2}\Big\langle \left[\begin{smallmatrix} \mathfrak{B}_1 \\ \mathfrak{B}_2 \end{smallmatrix}\right]e, \underbrace{\left(\left[\begin{smallmatrix} 0 & \mathrm{id}_{2k} \\ \mathrm{id}_{2k} & 0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right]^*\left[\begin{smallmatrix} 0 & \mathrm{id}_{2k} \\ \mathrm{id}_{2k} & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right]\right)}_{\leq 0}\left[\begin{smallmatrix} \mathfrak{B}_1 \\ \mathfrak{B}_2 \end{smallmatrix}\right]e\Big\rangle_{\mathbb{C}^{4k}} \leq 0.
\end{aligned}$$

This shows that M is dissipative. Maximal dissipativity of M now follows, by incorporating that S is a system node, from [PSR23, Prop. 3.8]. Furthermore, by combining the previously derived expression for $\Big\langle \left(\begin{smallmatrix} e \\ u \end{smallmatrix}\right), \left[\begin{smallmatrix} F\&G \\ -K\&L \end{smallmatrix}\right]\left(\begin{smallmatrix} e \\ u \end{smallmatrix}\right)\Big\rangle_{\mathcal{X}\times\mathbb{C}^m}$ with [PSR23, Thm. 3.11], we can establish that (37) holds. ❑

Remark 5.5. Systems defined by nodes of the form (35), where $\mathcal{H}$ is self-adjoint and positive and M in (36) is dissipative, are called *port-Hamiltonian* in [PSR23]. The framework there is more general, as $\mathcal{H}$ need not be bounded or boundedly invertible.

As a last result, we present a sufficient criterion for well-posedness.

Theorem 5.6. *Suppose that Assumption 2.1, 2.2, and 2.4 hold, and that the spatial domains are as in Section 2.1. Further assume the strict inequality*

$$W_B \begin{bmatrix} 0 & \mathrm{id}_{2k} \\ \mathrm{id}_{2k} & 0 \end{bmatrix} W_B^* > 0,$$

and let $W_{C,\mathrm{out}} \in \mathbb{C}^{p\times 4k}$*,* $p \in \mathbb{N}$*. Then for* $\mathcal{X}$ *as in (20b), and* $F\&G$*,* $K\&L$ *as in (34), and* $\mathcal{H}$ *as in (20a), the system corresponding to the node (35) is well-posed.*

Proof. It is not a loss of generality to assume that $m = 2k$, i.e., $W_{B,\mathrm{out}} = W_B$. In other words, all boundary conditions are actually ones in which the input acts. The case with the presence of homogeneous boundary conditions can be addressed by setting the corresponding inputs to zero.
We partition $W_B = [W_{B1}, W_{B2}]$ with $W_{B1}, W_{B2} \in \mathbb{C}^{2k\times 2k}$. Then (5) means that

$$W_{B1}W_{B2}^* + W_{B2}W_{B1}^* > 0. \tag{39}$$

Since for any $x \in \ker W_{B2}^*$, it holds that

$$x^*(W_{B1}W_{B2}^* + W_{B2}W_{B1}^*)x = 0,$$

(39) implies that $x = 0$. This shows that W_{B2} is invertible. Now we define

$$\widetilde{W_C} = [W_{B2}^{-*}, 0_{2k\times 2k}],$$

and consider the system node

$$\widetilde{S} := \left[\begin{smallmatrix} F\&G \\ \widetilde{K\&L} \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathcal{H} & 0 \\ 0 & \mathrm{id}_m \end{smallmatrix}\right]$$

with

$$\widetilde{K\&L}\begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{H} \\ \boldsymbol{V} \\ \boldsymbol{E} \\ u \end{pmatrix} = \widetilde{W_C}\begin{pmatrix} \boldsymbol{V}(0) \\ \boldsymbol{I}(0) \\ \boldsymbol{V}(1) \\ -\boldsymbol{I}(1) \end{pmatrix}.$$

Let $\mathfrak{B}_1$ be as in (27). Define the system

$$\begin{pmatrix} \dot{x}(t) \\ \widetilde{y}(t) \end{pmatrix} = \left[\begin{smallmatrix} F\&G \\ \widetilde{K\&L} \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathcal{H} & 0 \\ 0 & \mathrm{id}_m \end{smallmatrix}\right]\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \tag{40}$$

with output equation

$$\widetilde{y}(t) := W_{B2}^{-*}\mathfrak{B}_1\mathcal{H}x(t).$$

We have

$$\begin{aligned}
&\begin{bmatrix} W_B \\ \widetilde{W_C} \end{bmatrix}\begin{bmatrix} 0 & \mathrm{id}_{2k} \\ \mathrm{id}_{2k} & 0 \end{bmatrix}\begin{bmatrix} W_B^* & \widetilde{W_C}^* \end{bmatrix} = \begin{bmatrix} W_{B1} & W_{B2} \\ W_{B2}^{-*} & 0 \end{bmatrix}\begin{bmatrix} 0 & \mathrm{id}_{2k} \\ \mathrm{id}_{2k} & 0 \end{bmatrix}\begin{bmatrix} W_{B1}^* & W_{B2}^{-1} \\ W_{B2}^* & 0 \end{bmatrix} \\
&= \begin{bmatrix} W_{B1} & W_{B2} \\ W_{B2}^{-*} & 0 \end{bmatrix}\begin{bmatrix} 0 & \mathrm{id}_{2k} \\ \mathrm{id}_{2k} & 0 \end{bmatrix}\begin{bmatrix} W_{B2}^* & 0 \\ W_{B1}^* & W_{B2}^{-1} \end{bmatrix} = \begin{bmatrix} W_{B1}W_{B2}^* + W_{B2}W_{B1}^* & \mathrm{id}_{2k} \\ \mathrm{id}_{2k} & 0 \end{bmatrix}.
\end{aligned}$$

Let

$$W := W_{B2}^{-*}W_{B1}^* + W_{B1}^*W_{B2}^{-1} = W_{B2}^{-1}(W_{B1}W_{B2}^* + W_{B2}W_{B1}^*)W_{B2}^{-*} > 0,$$

and define $\delta > 0$ as be the minimal eigenvalue of W. Then Theorem 5.4 gives rise to the energy balance

$$\begin{aligned}
&\tfrac{1}{2}\langle x(t), \mathcal{H}x(t)\rangle_{\mathcal{X}} - \tfrac{1}{2}\langle x(0), \mathcal{H}x(0)\rangle_{\mathcal{X}} \\
&= \int_0^t \mathrm{Re}\langle u(\tau), \widetilde{y}(\tau)\rangle_{\mathbb{C}^m}\,\mathrm{d}\tau + \int_0^t \mathrm{Re}\langle \mathcal{H}x(\tau), \mathfrak{R}\mathcal{H}x(\tau)\rangle_{\mathcal{X}}\,\mathrm{d}\tau - \frac{1}{2}\int_0^t \langle \widetilde{y}(\tau), W\widetilde{y}(\tau)\rangle_{\mathbb{C}^{2k}}\,\mathrm{d}\tau \\
&\le \int_0^t \mathrm{Re}\langle u(\tau), \widetilde{y}(\tau)\rangle_{\mathbb{C}^m} - \delta\|\widetilde{y}(\tau)\|_{\mathbb{C}^m}^2\,\mathrm{d}\tau
\end{aligned}$$

Now using that

$$\mathrm{Re}\langle u(\tau), \widetilde{y}(\tau)\rangle_{\mathbb{C}^m} \le \tfrac{1}{2\delta}\|u(\tau)\|_{\mathbb{C}^m}^2 + \tfrac{\delta}{2}\|\widetilde{y}(\tau)\|_{\mathbb{C}^m}^2,$$

we obtain

$$\tfrac{1}{2}\langle x(t), \mathcal{H}x(t)\rangle_{\mathcal{X}} - \tfrac{1}{2}\langle x(0), \mathcal{H}x(0)\rangle_{\mathcal{X}} \le \tfrac{1}{2\delta}\|u\|_{\mathrm{L}^2((0,t);\mathbb{C}^k)}^2 - \tfrac{\delta}{2}\|\widetilde{y}\|_{\mathrm{L}^2((0,t);\mathbb{C}^k)}^2,$$

and thus

$$\langle x(t), \mathcal{H}x(t)\rangle_{\mathcal{X}} + \delta\|\widetilde{y}\|_{\mathrm{L}^2((0,t);\mathbb{C}^k)}^2 \le \langle x(0), \mathcal{H}x(0)\rangle_{\mathcal{X}} + \tfrac{1}{\delta}\|u\|_{\mathrm{L}^2((0,t);\mathbb{C}^k)}.$$

Now using the equivalence of the standard norm in $\mathcal{X}$ and the one in (30), we can immediately conclude that (40) is a well-posed system. That is, there exists some $c > 0$, such that

$$\|x(t)\|_{\mathcal{X}} + \|\widetilde{y}\|_{\mathrm{L}^2((0,t);\mathbb{C}^k)} \le c\big(\|x(0)\|_{\mathcal{X}} + \|u\|_{\mathrm{L}^2((0,t);\mathbb{C}^k)}\big).$$

Now we show well-posedness of the actual system. It follows directly from construction that $\left[\begin{smallmatrix} W_B \\ \widetilde{W_C} \end{smallmatrix}\right]$ is invertible. Then we have, for almost all $t \in [0, T]$,

$$y(t) \;=\; W_C\left[\begin{smallmatrix} \mathfrak{B}_1 \\ \mathfrak{B}_2 \end{smallmatrix}\right]\mathcal{H}x(\tau) \;=\; W_C\left[\begin{smallmatrix} W_B \\ \widetilde{W_C} \end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix} W_B \\ \widetilde{W_C} \end{smallmatrix}\right]\mathcal{H}x(\tau) \;=\; W_C\left[\begin{smallmatrix} W_B \\ \widetilde{W_C} \end{smallmatrix}\right]^{-1}\left(\begin{smallmatrix} u(t) \\ \widetilde{y}(t) \end{smallmatrix}\right).$$

Now, by choosing

$$\gamma := \left\| W_C\left[\begin{smallmatrix} W_B \\ \widetilde{W_C} \end{smallmatrix}\right]^{-1} \right\|,$$

we obtain

$$\begin{aligned}
\|x(t)\|_{\mathcal{X}} + \|y\|_{\mathrm{L}^2((0,t);\mathbb{C}^k)} &\le \|x(t)\|_{\mathcal{X}} + \gamma\|y\|_{\mathrm{L}^2((0,t);\mathbb{C}^k)} + \gamma\|u\|_{\mathrm{L}^2((0,t);\mathbb{C}^k)} \\
&\le c\max\{1,\gamma\}\big(\|x(0)\|_{\mathcal{X}} + \|u\|_{\mathrm{L}^2((0,t);\mathbb{C}^k)}\big) + \gamma\|u\|_{\mathrm{L}^2((0,t);\mathbb{C}^k)}.
\end{aligned}$$

Consequently, the generalized trajectories fulfill (33) for

$$c_t = \max\{1, c\}\max\{1, \gamma\}(1 + \gamma),$$

which completes the proof. ❑

6. Conclusion

We have presented an analysis for radiating cable harnesses, resulting in telegrapher's and Maxwell's equations that are coupled through boundary conditions. An operator and systems theoretic analysis of the entire coupled system has been conducted using the theories of boundary triplesand system nodes. It has been shown that the autonomous part is described by a strongly continuous semigroup, and a sufficient criterion for well-posedness has been provided.

Acknowledgments.
This work was supported by the collaborative research center SFB 1701 "Port-Hamiltonian Systems", funded by the German Research Foundation (DFG), project number 531152215.

Appendix A. Boundary triples

We review the most important properties of boundary triples for skew-symmetric operators for this work. More details can be found in [GG91, Chap. 3] and [BHdS20], where boundary triples are regarded for symmetric operators. For skew-symmetric operators we refer to [Skr21a] (we will use the skew-symmetric version).

A *linear relation* T from a space $\mathcal{X}$ to a space $\mathcal{Y}$ is a subspace of $\mathcal{X} \times \mathcal{Y}$. Clearly, by an identification with its graph, any linear operator is also a linear relation. In this context, linear relations can be regarded as multi-valued linear operators.

Definition A.1. A linear relation T on a Hilbert space $\mathcal{X}$ (from $\mathcal{X}$ to $\mathcal{X}$)

- is *dissipative*, if $\operatorname{Re}\langle x, y\rangle_{\mathcal{X}} \leq 0$ for every $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in T$ and
- *maximally dissipative*, if additionally it has no proper dissipative extension.

Definition A.2. Let $\mathfrak{A}_0\colon \operatorname{dom}(\mathfrak{A}_0) \subseteq \mathcal{X} \to \mathcal{X}$ be a densely defined, skew-symmetric, and closed operator on a Hilbert space $\mathcal{X}$. By a *boundary triple* for $\mathfrak{A}_0^*$ we mean a triple $(\mathcal{U}, \mathfrak{B}_1, \mathfrak{B}_2)$ consisting of a Hilbert space $\mathcal{U}$, and two linear operators $\mathfrak{B}_1, \mathfrak{B}_2\colon \operatorname{dom}\mathfrak{A}_0^* \to \mathcal{U}$ such that

(i) the mapping $\left(\begin{smallmatrix} \mathfrak{B}_1 \\ \mathfrak{B}_2 \end{smallmatrix}\right)\colon \operatorname{dom}\mathfrak{A}_0^* \to \mathcal{U} \times \mathcal{U}, x \mapsto \left(\begin{smallmatrix} \mathfrak{B}_1 x \\ \mathfrak{B}_2 x \end{smallmatrix}\right)$ is surjective, and
(ii) for $x, y \in \operatorname{dom}\mathfrak{A}_0^*$ there holds

$$\langle \mathfrak{A}_0^* x, y\rangle_X + \langle x, \mathfrak{A}_0^* y\rangle_X = \langle \mathfrak{B}_1 x, \mathfrak{B}_2 y\rangle_{\mathcal{U}} + \langle \mathfrak{B}_2 x, \mathfrak{B}_1 y\rangle_{\mathcal{U}}. \tag{41}$$

An important result for boundary triples is that maximally dissipative restrictions of $\mathfrak{A}_0^*$ can be charcaterized by maximally dissipative linear relations Θ on $\mathcal{U}$. The following theorem from [Skr21a, Prop. 2.4.10] will clarify that.

Theorem A.3. *Let $(\mathcal{U}, \mathfrak{B}_1, \mathfrak{B}_2)$ be a boundary triple for $\mathfrak{A}_0^*$ and Θ be a maximally dissipative linear relation on $\mathcal{U}$. Then $\mathfrak{A}_\Theta$, which is the restriction of $\mathfrak{A}_0^*$ to*

$$\operatorname{dom}\mathfrak{A}_\Theta = \left\{ x \in \operatorname{dom}\mathfrak{A}_0^* \,\middle|\, \left(\begin{smallmatrix} \mathfrak{B}_1 x \\ \mathfrak{B}_2 x \end{smallmatrix}\right) \in \Theta \right\}$$

is a maximally dissipative operator.

Since the systems considered in this article has finite-dimensional input and output spaces, our attention is directed towards linear relations in the form of $\Theta = \ker \begin{bmatrix} W_1 & W_2 \end{bmatrix}$, where W_1 and W_2 are square matrices. We establish a criterion for maximal dissipativity of such relations.

Lemma A.4. *Let $W_1, W_2 \in \mathbb{C}^{\ell \times \ell}$. Then, for $W := \begin{bmatrix} W_1 & W_2 \end{bmatrix}$, the relation $\ker W$ is maximally dissipative, if W has full row rank and $W_1 W_2^* + W_2 W_1^* \geq 0$.*

Proof. The relation $\operatorname{ran} \left[\begin{smallmatrix} W_2^* \\ -W_1^* \end{smallmatrix}\right]$ is maximally dissipative by [GHR21, Lem. 3.5] and, further, it is, in the sense of [BHdS20, Def. 1.3.1], the adjoint of $\ker W$. Then we can conclude from [BHdS20, Prop. 1.6.7] that $\ker W$ is maximally dissipative. ❑

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